Computing persistence for simplicial maps with application to data sparsification

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Joint work with Fengtao Fan and Yusu Wang

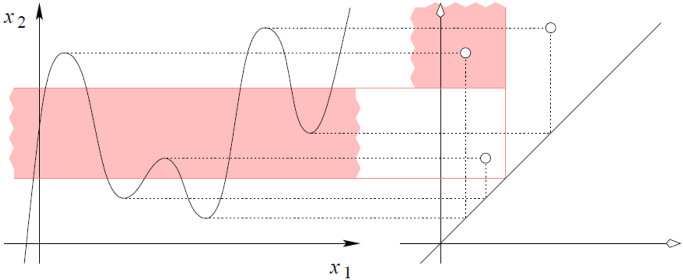
Topological Persistence

- Persistent homology [ELZ 2000] [under inclusion maps]

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq K_4 \subseteq \ldots \subseteq K_n$$

$$\mathcal{M}: H(K_1) \longrightarrow H(K_2) \longrightarrow H(K_3) \longrightarrow H(K_4) \longrightarrow \ldots \longrightarrow H(K_n)$$

- Persistent diagram $\mathcal{DM}$ [CEH 2006]



Topological Persistence

- Persistent homology [ZC05] [under simplicial maps]

$$K_1 \xrightarrow{f_1} K_2 \xrightarrow{f_2} K_3 \xrightarrow{f_3} K_4 \xrightarrow{f_4} \dots \xrightarrow{f_{n-1}} K_n$$

$$\mathcal{M}: \; H(K_1) \xrightarrow{f_{1*}} H(K_2) \xrightarrow{f_{2*}} H(K_3) \xrightarrow{f_{3*}} H(K_4) \xrightarrow{f_{4*}} \dots \xrightarrow{f_{n-1*}} H(K_n)$$

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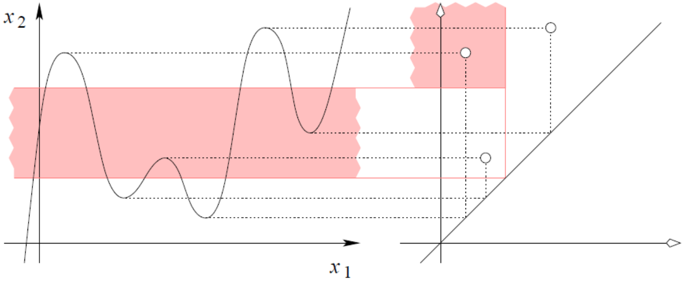
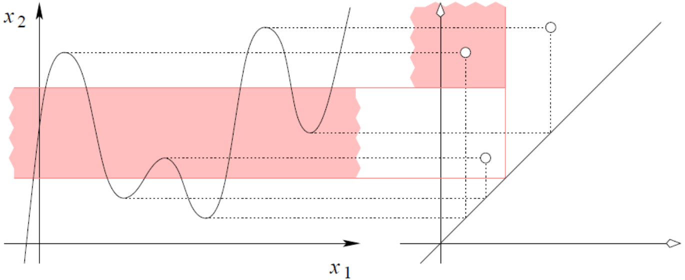


Topological Persistence

- Zigzag persistent homology [CS10]

$$\begin{array}{ccccccccccccc} & K_1 & \subseteq & K_2 & \supseteq & K_3 & \subseteq & K_4 & \supseteq & \dots & \subseteq & K_n \\ \mathcal{Z}: & H(K_1) & \longrightarrow & H(K_2) & \longleftarrow & H(K_3) & \longrightarrow & H(K_4) & \longleftarrow & \dots & \longrightarrow & H(K_n) \end{array}$$

- Persistent diagram $\mathcal{DZ}$



Algorithms for topological persistence

- Monotone persistence under inclusion maps [ELZ00]

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Algorithms for topological persistence

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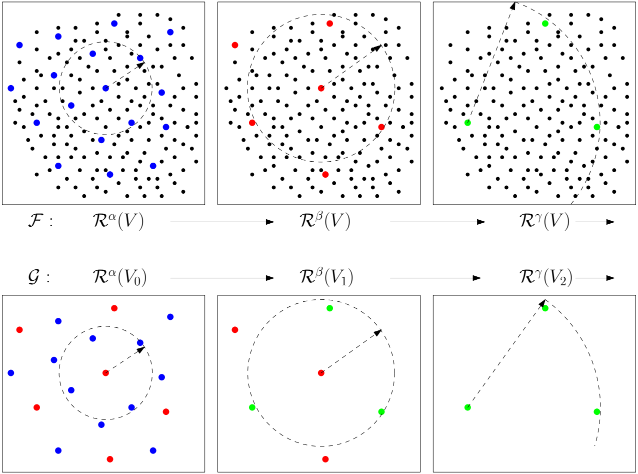
This paper [DFW14]:

- A simple algorithm for zigzag persistence under simplicial maps
- More efficient algorithm for monotone persistence under simplicial maps

An Application

Sparsified Rips complexes [Sheehy 12]

- $\mathcal{R}^\alpha(V)$: Rips complex on point set V with parameter α;



- Goal: approximate $\mathcal{DF}$ by $\mathcal{DG}$ [Sheehy 12];

Persistence diagram of filtered Rips complexes

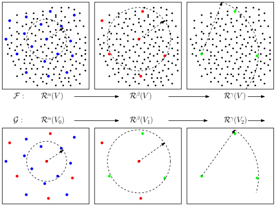
- Filtration of Rips complexes;

$$\mathcal{R}^{\alpha}(V) \hookrightarrow \mathcal{R}^{\alpha(1+\epsilon)}(V) \hookrightarrow \mathcal{R}^{\alpha(1+\epsilon)^2}(V) \hookrightarrow \cdots \hookrightarrow \mathcal{R}^{\alpha(1+\epsilon)^m}(V)$$

- The persistent module $\mathcal{F}$

$$\mathcal{F} : H_*(\mathcal{R}^{\alpha}(V)) \xrightarrow{i_*} H_*(\mathcal{R}^{\alpha(1+\epsilon)}(V)) \xrightarrow{i_*} \cdots \xrightarrow{i_*} H_*(\mathcal{R}^{\alpha(1+\epsilon)^m}(V))$$

Vertex maps through sequence of subsamples



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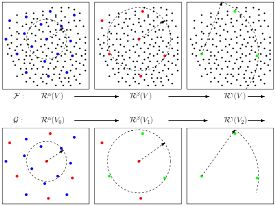

- Given $V_0 := V$, construct point sets V_k, $k = 0, 1, ..., m$, where V_{k+1} is a $\frac{1}{2}\alpha\epsilon^2(1+\epsilon)^{k-1}$-sampling of V_k.

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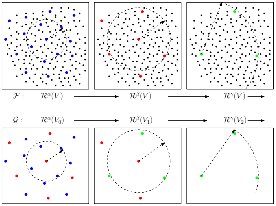

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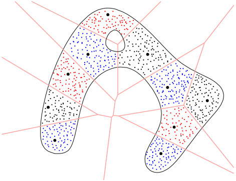

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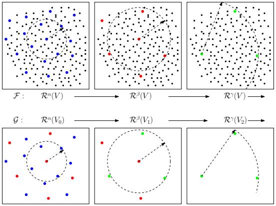

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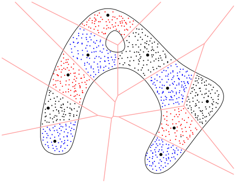

- $\pi_k : V_k \to V_{k+1}$, where $\pi_k(v)$ is the closest point in V_{k+1} to v;

$$V_0 \xrightarrow{\pi_0} V_1 \xrightarrow{\pi_1} V_2 \xrightarrow{\pi_2} V_3 \to \ldots \to V_m$$

Approximating persistence diagram

- π_k induces a simplicial map h_k,

$$h_k : \mathcal{R}^{\alpha(1+\epsilon)^k}(V_k) \to \mathcal{R}^{\alpha(1+\epsilon)^{k+1}}(V_{k+1})$$

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- A sequence of simplicial maps

$$\mathcal{R}^{\alpha}(V_0) \xrightarrow{h_0} \mathcal{R}^{\alpha(1+\epsilon)}(V_1) \xrightarrow{h_1} \mathcal{R}^{\alpha(1+\epsilon)^2}(V_2) \xrightarrow{h_2} \cdots \xrightarrow{h_{m-1}} \mathcal{R}^{\alpha(1+\epsilon)^m}(V_m)$$

provides a persistent module $\mathcal{G}$

$$\mathcal{G} : H_*(\mathcal{R}^{\alpha}(V_0)) \xrightarrow{h_{0*}} H_*(\mathcal{R}^{\alpha(1+\epsilon)}(V_1)) \xrightarrow{h_{1*}} \cdots \xrightarrow{h_{m-1*}} H_*(\mathcal{R}^{\alpha(1+\epsilon)^m}(V))$$

Proposition (DFW13)

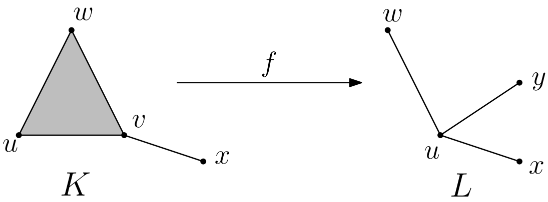
Interleaving of $\mathcal{F}$ and $\mathcal{G}$ provides $d_B(\mathcal{D}\mathcal{G}, \mathcal{D}\mathcal{F}) \leq 2log(1+\epsilon)$ in log-scale [CCGGO09].

Simplicial Maps

Basic Definitions

Definition

Simplicial map: A map $f : K \to L$ is simplicial if vertices for every simplex $\sigma \in K$ are mapped to the vertices of the simplex $f(\sigma)$ in L.



Basic definitions

Definition

Elementary simplicial map: $f : K \to K'$ is *elementary* if the vertex map f_v is identical everywhere except possibly on a set $X \subseteq V(K)$ for which $f_v(X)$ is a single vertex in K'.

- If $X = \emptyset$ and $K' \setminus K$ is a single simplex, f is an elementary inclusion;
- If $|X| = 2$ and f surjective, *elementary collapse*;

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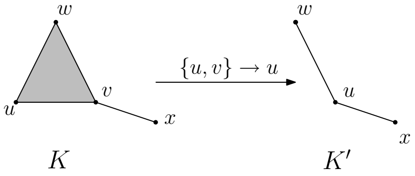
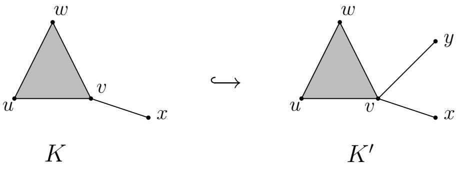


Elementary collapse ;



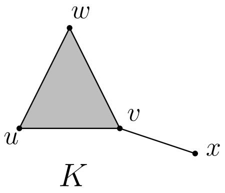
Elementary inclusion;

Basic definitions

Definition

Stars and Links of $X \subseteq K$:

$$\begin{aligned}
\texttt{St}X &:= \{\sigma \mid \sigma \text{ is a coface of a simplex in } X\} \\
\overline{\texttt{St}}X &:= \{\text{all faces of simplices in } \texttt{St}X\} \\
\texttt{Lk}X &:= \overline{\texttt{St}}X - \texttt{St}\overline{X}
\end{aligned}$$



$\overline{\texttt{St}}v = \{v, vw, vx, uv, uvw, u, w, x$

$\texttt{Lk}v = \{u, w, x, uw\}, \texttt{Lk}uv = \{w\}$

Decomposition of simplicial maps

Proposition

If $f : K \to K'$ is a simplicial map, then there are elementary inclusions and collapses f_i

$$K \xrightarrow{f_1} K_1 \xrightarrow{f_2} K_2 \cdots \xrightarrow{f_n} K_n = K'$$

so that $f = f_n \circ \cdots \circ f_2 \circ f_1$.

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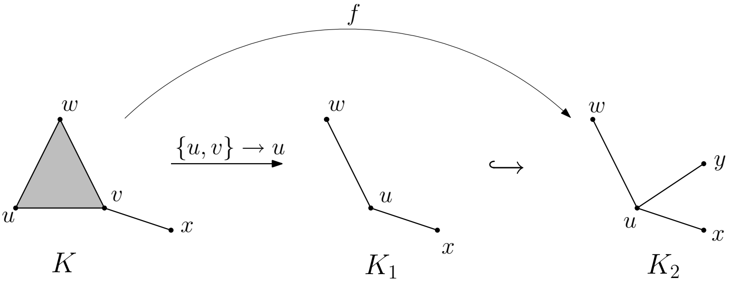


Zigzag simplicial maps

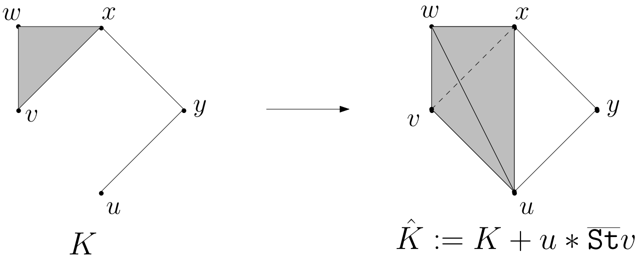
$$K_1 \xrightarrow{f_1} K_2 \xleftarrow{f_2} K_3 \xrightarrow{f_3} K_4 \xleftarrow{f_4} \ldots \xrightarrow{f_{n-1}} K_n$$

Simulating elementary maps by inclusions

- If $f : K \to K'$ elementary inclusion
  - no action;
- If $f : K \to K'$ elementary collapse

$$f(\{u, v\}) \longrightarrow u \in K'$$

  - Augmenting K as $\widehat{K} := K + u * \overline{\mathtt{St}}v$;
  - $*$ is the join (coning) operation;



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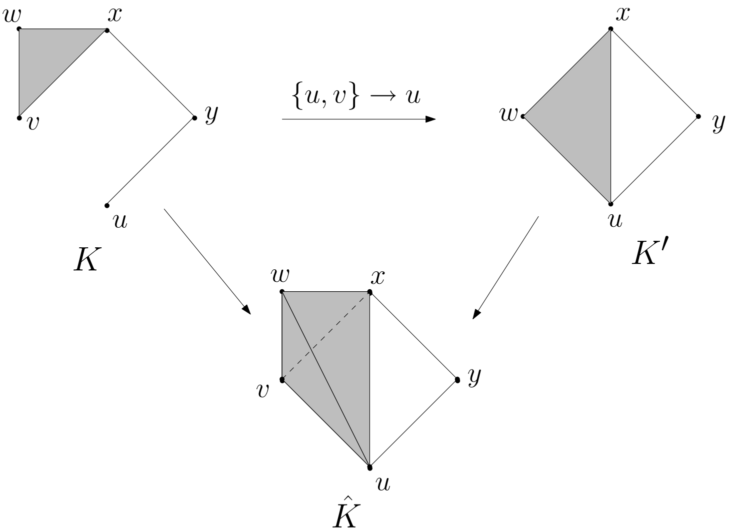
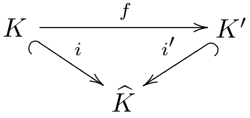
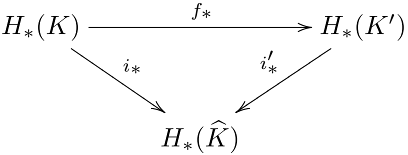


Simulating elementary maps by inclusions

- Consider the simplicial map f and inclusions together:





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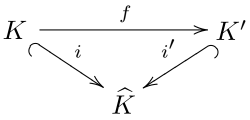

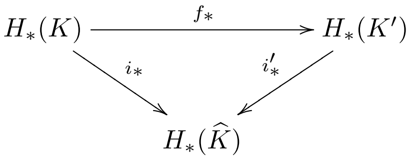

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- i'_* *is an isomorphism;*

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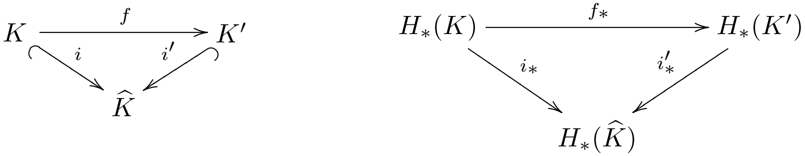

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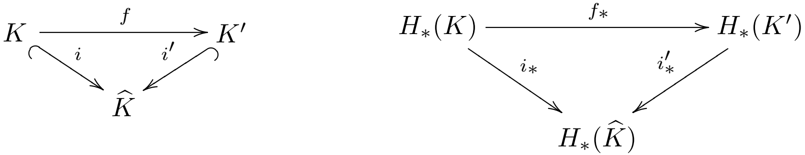


Proposition

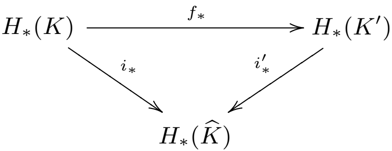
- *The right diagram is commutative;*
- $i^{'}_{*}$ *is an isomorphism;*

- $i^{'} \circ f$ and i are contiguous : $i' \circ f(\sigma) \cup \sigma$ is a simplex in $\widehat{K}$;
- Projection $\pi : \widehat{K} \to K'$ induced by vertex map is homotopy inverse of i'

$$\pi(p) = \begin{cases} u & \text{if } p = v \\ p & \text{otherwise} \end{cases}$$

Persistence for simplicial maps

- i'_* is an isomorphism ;
- $f_* = i_* \circ (i'_*)^{-1}$;



- The following 3 sequences are equivalent:

$$H_*(K) \xrightarrow{f_*} H_*(K')$$

$$H_*(K) \xrightarrow{i_*} H_*(\widehat{K}) \xrightarrow{(i'_*)^{-1}} H_*(K')$$

$$H_*(K) \xrightarrow{i_*} H_*(\widehat{K}) \xleftarrow{i'_*} H_*(K')$$

Persistence for simplicial maps

- $i_*^{'}$ is an isomorphism ;
- $f_* = i_* \circ (i_*^{'})^{-1}$;

$$\begin{array}{ccc} H_*(K) & \xrightarrow{\quad f_* \quad} & H_*(K') \\ & {\scriptstyle i_*} \searrow \quad \swarrow {\scriptstyle i'_*} & \\ & H_*(\widehat{K}) & \end{array}$$

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Proposition

Persistence of $K_1 \xrightarrow{f_1} K_2 \xrightarrow{f_2} K_3 \xrightarrow{f_3} K_4 \xrightarrow{f_4} \ldots \xrightarrow{f_{n-1}} K_n$ *is captured by:*

$$\mathcal{Z} : H_*(K_1) \xhookrightarrow{i_*} H_*(\widehat{K_1}) \xhookleftarrow{i'_*} H_*(K_2) \xhookrightarrow{i_*} H_*(\widehat{K_2}) \xhookleftarrow{i'_*} H_*(K_3) \ldots \xhookleftarrow{i'_*} H_*(K_n)$$

corresponding to the zigzag sequence :

$$K_1 \xhookrightarrow{i_1} \widehat{K_1} \xhookleftarrow{i'_1} K_2 \xhookrightarrow{i_2} \widehat{K_2} \xhookleftarrow{i'_2} K_3 \ldots \xhookleftarrow{i'_{n-1}} K_n$$

Zigzag persistence of simplicial maps

Proposition

For a zigzag sequence of elementary simplicial maps,

$$K_1 \xrightarrow{f_1} K_2 \xleftarrow{f_2} K_3 \xrightarrow{f_3} \ldots \xrightarrow{f_{n-1}} K_n$$

one can compute its zigzag persistence through zigzag filtration:

$$K_1 \xhookrightarrow{i_1} \widehat{K_1} \xhookleftarrow{i'_1} K_2 \xhookrightarrow{i'_2} \widehat{K_3} \xhookleftarrow{i_2} K_3 \xhookrightarrow{i_3} \ldots \xhookleftarrow{i'_{n-1}} K_n$$

- $(i'_k)_*$'s are isomorphisms;
- $(i_k)_* = (i'_k)_* \circ (f_k)_*$;
- The following two zigzag modules have the same persistence diagram:

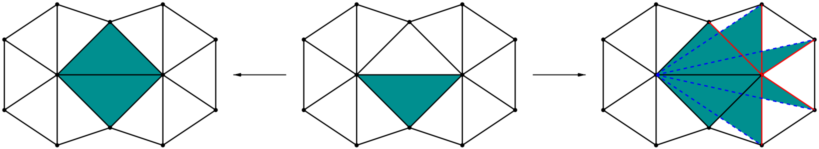
$$H_*(K_1) \xrightarrow{f_{1*}} H_*(K_2) \xleftarrow{f_{2*}} H_*(K_3) \xrightarrow{f_{3*}} \cdots \rightarrow H_*(K_n)$$

$$H_*(K_1) \xrightarrow{i_*} H_*(\widehat{K_1}) \simeq H_*(K_2) \simeq H_*(\widehat{K_3}) \xleftarrow{i_*} H_*(K_3) \xrightarrow{i_*} \cdots \xleftarrow{i_*} H_*(K_n)$$

Annotations

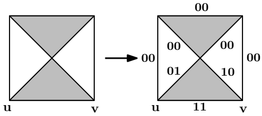
Persistence using annotation

- Persistence of non-zigzag sequence of simplicial maps by annotations
  - Maintains a consistent cohomology basis;
  - Cohomology basis elements are time stamped for tracking persistence;
  - Creates much less intermediate simplices;

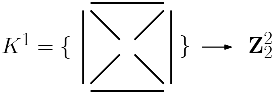


Annotation $\mathbf{a}$ for simplicial complex K

- Mapping $\mathbf{a} : K^p \rightarrow \mathbb{Z}_2^g$
  - K^p : the set of p-simplices of K;
  - $\mathbf{a}_\sigma = \mathbf{a}(\sigma)$: a binary vector of length g;



- $\mathbf{a}$ valid if
  - $g = \mathrm{rank} H_p(K)$;
  - $\mathbf{a}_{z_1} = \mathbf{a}_{z_2}$ iff $[z_1] = [z_2]$;



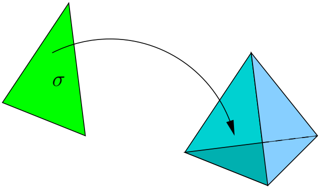
Proposition

The following two statements are equivalent:

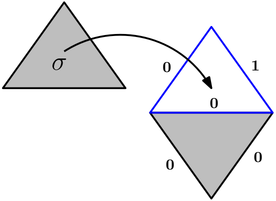
1. *An annotation $\mathbf{a} : K^p \rightarrow \mathbb{Z}_2^g$ is valid*
2. *The cochains $\{\phi_i\}_{i=1,\cdots,g}$ given by $\phi_i(\sigma) = \mathbf{a}_\sigma[i]$ for all $\sigma \in K^p$ are cocycles whose cohomology classes constitute a basis of $H^p(K)$.*

Elementary inclusion [SMV-J 2011]

- Elementary inclusion
  - Obtain a valid annotation of K_{i+1} from K_i after inserting p-simplex $\sigma = K_{i+1} \setminus K_i$;
  - Two cases : $\mathbf{a}_{\partial\sigma} = \mathbf{0}$ or $\mathbf{a}_{\partial\sigma} \neq \mathbf{0}$;



$\mathbf{a}_{\partial\sigma} = \mathbf{0}$;



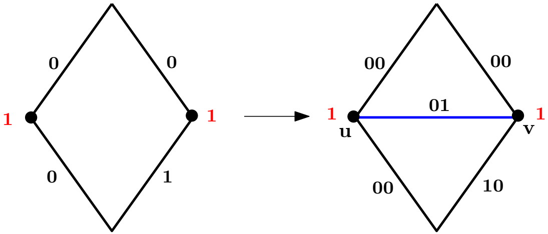
$\mathbf{a}_{\partial\sigma} \neq \mathbf{0}$;

Elementary inclusion case 1

- Elementary inclusion case 1 : $\mathbf{a}_{\partial\sigma} = \mathbf{0}$
  - σ creates a p-cycle in K_{i+1};
  - Augment the annotation $[b_1, b_2, \ldots, b_g]$ for p-simplex τ to $[b_1, b_2, \ldots, b_g, b_{g+1}]$

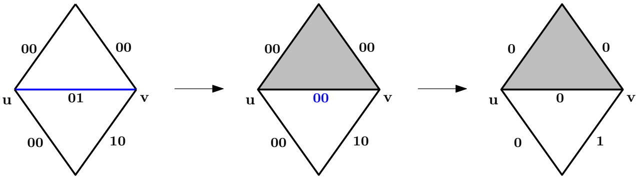
$$b_{g+1} = \begin{cases} 0 & \text{if } \tau \neq \sigma \\ 1 & \text{if } \tau = \sigma \end{cases}$$

  - The new element time stamped as $i + 1$;



Elementary inclusion case 2

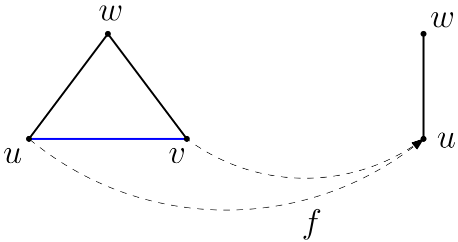
- Elementary inclusion case 2 : $\mathbf{a}_{\partial\sigma} \neq \mathbf{0}$
  - σ kills a $(p-1)$-cycle $\partial\sigma$;
  - b_u the last nonzero element in $\mathbf{a}_{\partial\sigma} = [b_1, b_2, \ldots, b_u, \ldots, b_g]$;
  - $\mathbf{a}_\tau = \mathbf{a}_\tau + \mathbf{a}_{\partial\sigma}$ if $(\mathbf{a}_\tau)_u = 1$;
  - Remove u-th element from all annotations;



Elementary collapse

Definition

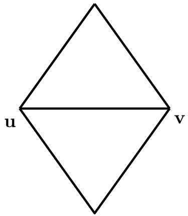
For an elementary collapse $f_i : K_i \rightarrow K_{i+1}$, a simplex $\sigma \in K_i$ is called vanishing if the cardinality of $f_i(\sigma)$ is one less than that of σ. Two simplices σ and σ' are called mirror of each other if one adjoins u and the other v, and share rest of the vertices.



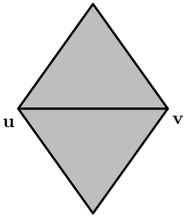
- uv vanishing simplex;
- wu and wv mirror to each other;

Elementary collapse

- Elementary collapse
  - Obtain a valid annotation of K_{i+1} from K_i after collapsing (u, v) to u;
  - Two cases : (u, v) satisfies or not link condition $\mathtt{Lk}uv = \mathtt{Lk}u \cap \mathtt{Lk}v$.



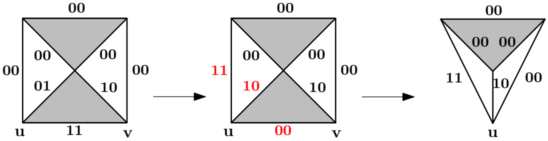
uv does not satisfy link condition;



uv satisfies link condition;

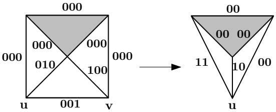
Elementary collapse case 1

- Elementary collapse case 1 : (u, v) satisfying link condition
  - τ any mirror simplex containing u;
  - σ unique coface of τ containing the edge uv;
  - $\mathbf{a}_\omega = \mathbf{a}_\omega + \mathbf{a}_\sigma$ for any coface ω of τ of codimension one

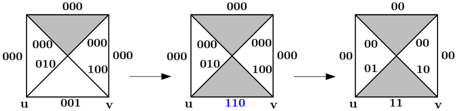


Elementary collapse case 2

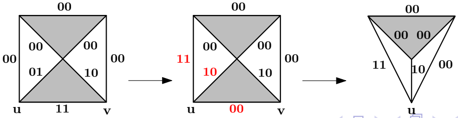
- Elementary collapse case 2 : (u, v) not satisfying link condition
  - Insert necessary simplices σ to meet the link condition ;
  - Apply the elementary collapse case 1;
- Collapsing edge uv:



1. Inclusion :



2. Elementary collapse case 1:



Data Sparsification

Sparsified complexes

- Sparsified Vietoris-Rips complex [Sheehy 12]
- Graph Induced Complex (GIC) [DFW13]

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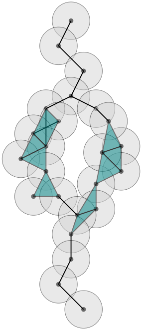

$\mathcal{R}^{\alpha}(V)$

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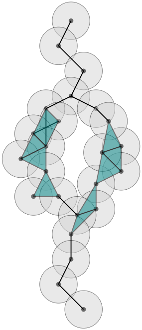

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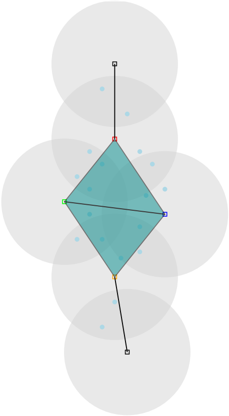

$\mathcal{R}^{\beta}(V')$

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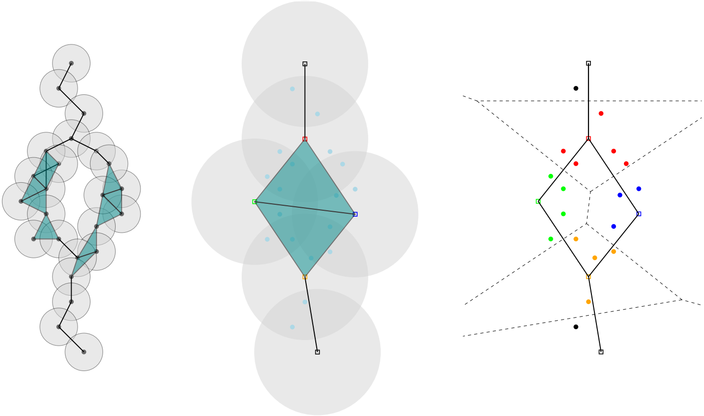
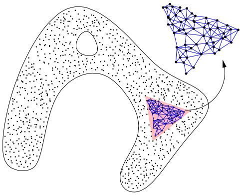


$\mathcal{R}^\alpha(V)$ $\mathcal{R}^\beta(V')$ $\mathcal{G}^\alpha(V, V')$

Graph Induced Complex



- V finite point set;
- $G(V)$ a graph;

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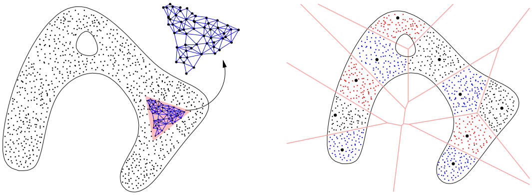

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- $V' \subset V$ a subset;
- $\nu : V \to V'$ surjective vertex map such that $\nu|_{V'} = \mathrm{id}$;

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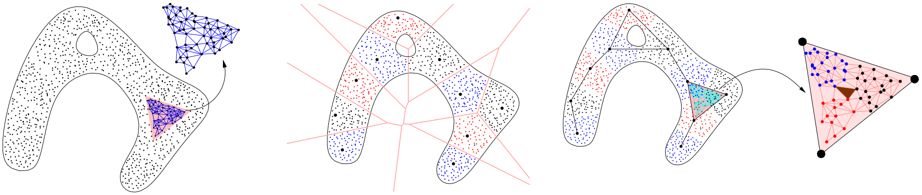

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- Graph induced complex $\mathcal{G}(V, V', \nu)$:

  $\sigma = \{v'_1, \dots, v'_{k+1}\}$, $v'_i \in V'$
  - a $(k+1)$-clique in $G(V)$ with vertices $v_1, \dots, v_{k+1}$;
  - $\nu(v_i) = v'_i$;

Approximating persistence diagram

- Rips filtration:

$$\mathcal{R}^{\alpha}(V) \hookrightarrow \mathcal{R}^{\alpha(1+\epsilon)}(V) \hookrightarrow \mathcal{R}^{\alpha(1+\epsilon)^2}(V) \hookrightarrow \cdots \hookrightarrow \mathcal{R}^{\alpha(1+\epsilon)^m}(V)$$

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- π_k induces a simplicial map f_k

$$f_k : \mathcal{G}^{\alpha(1+\epsilon)^{k-1}}(V_0, V_k) \to \mathcal{G}^{\alpha(1+\epsilon)^k}(V_0, V_{k+1})$$

Approximating persistence diagram

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Proposition (DFW14)

Interleaving of $\mathcal{F}$ and $\mathcal{L}$ provides $d_B(\mathcal{DF}, \mathcal{DL}) \leq 2(log(1+\epsilon))$ in log-scale [CCGGO09].

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- Other applications of simplicial maps?

Thank you !

Questions ?