Approximating Loops in a Shortest Homology Basis from Point Data

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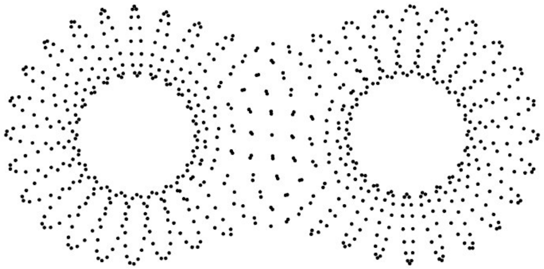
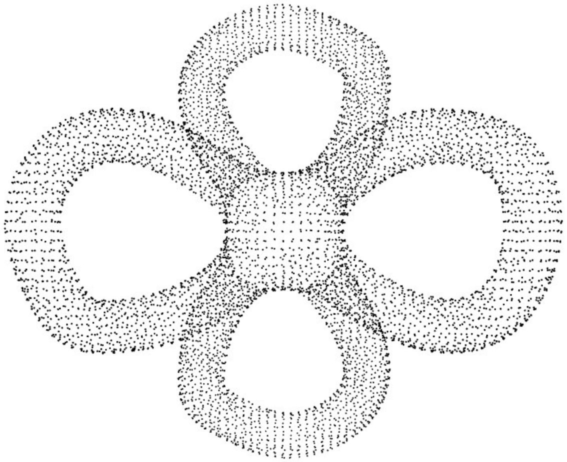
Joint work with Jian Sun and Yusu Wang

Our Goal





Point cloud

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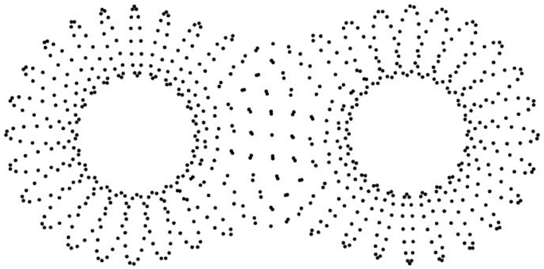

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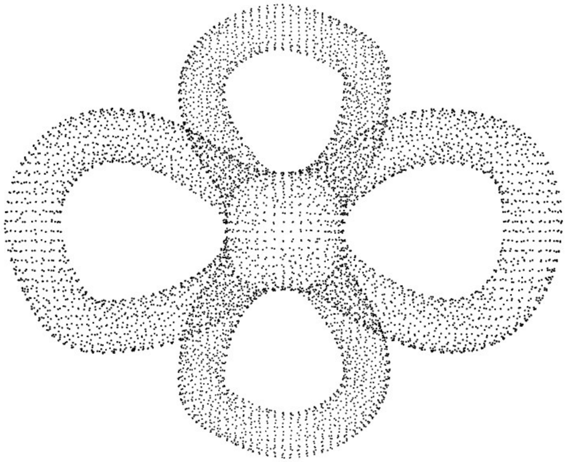

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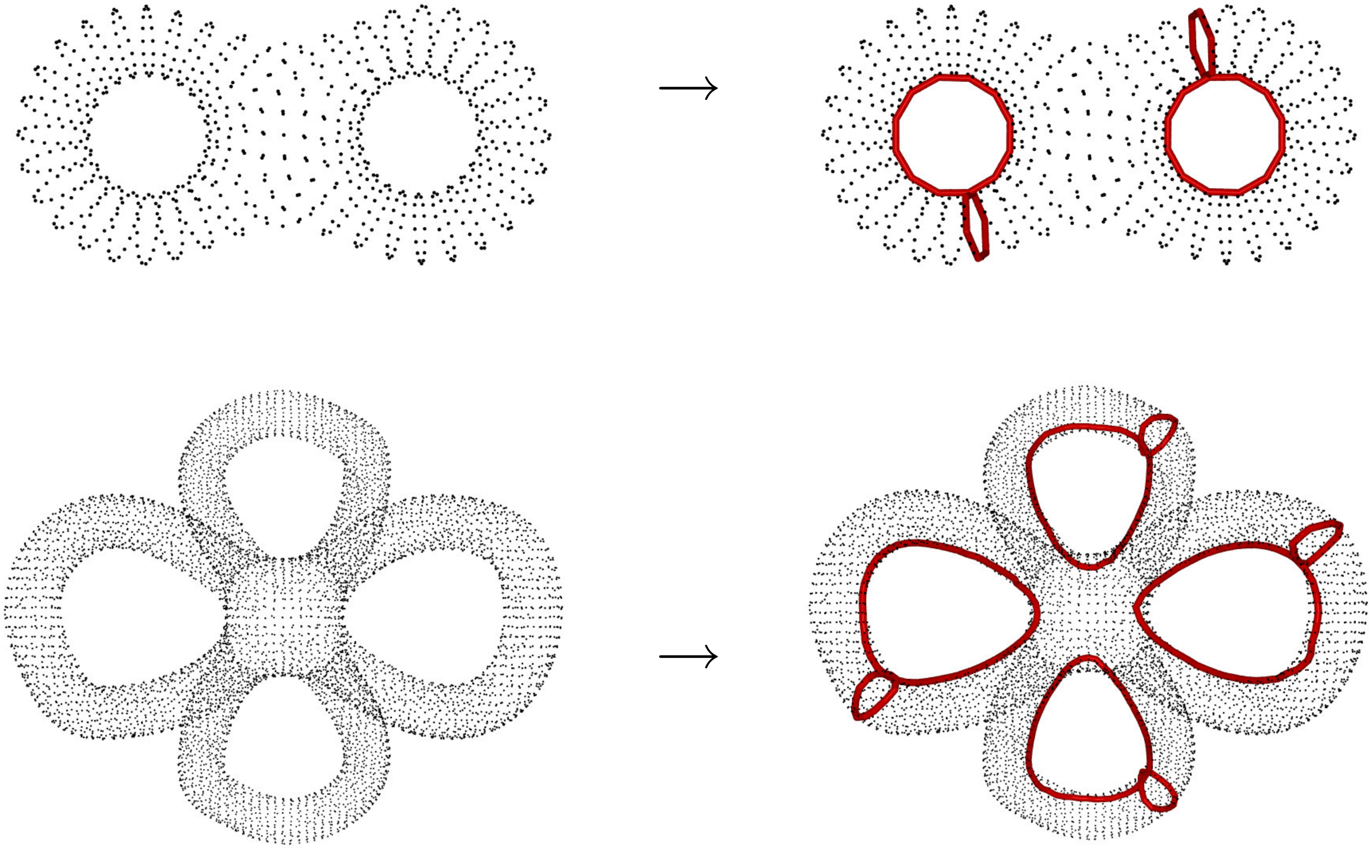

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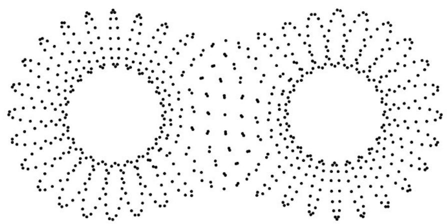

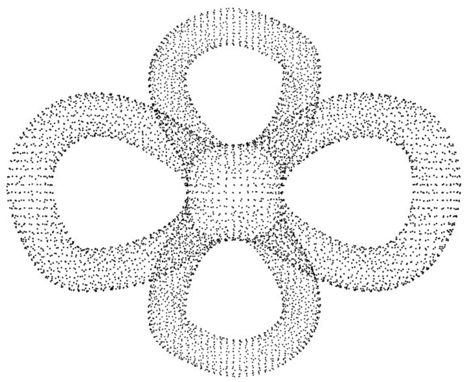

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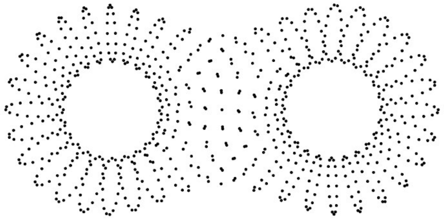

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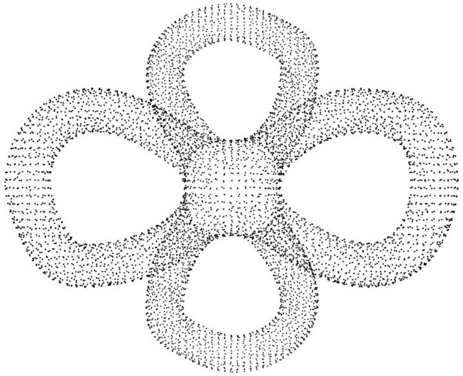

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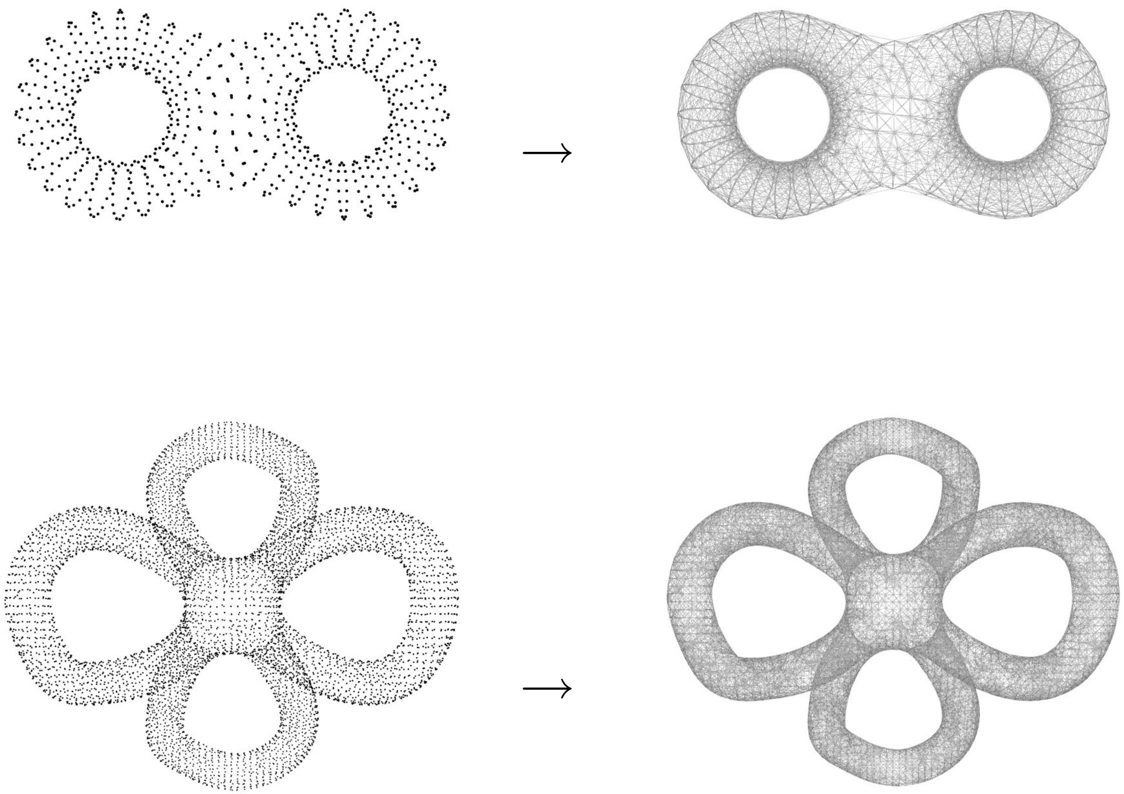

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Rips complex

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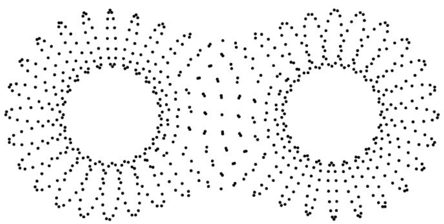
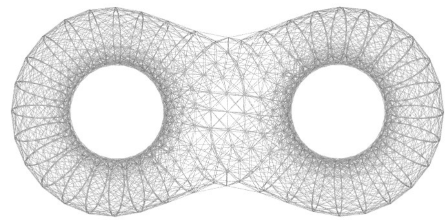
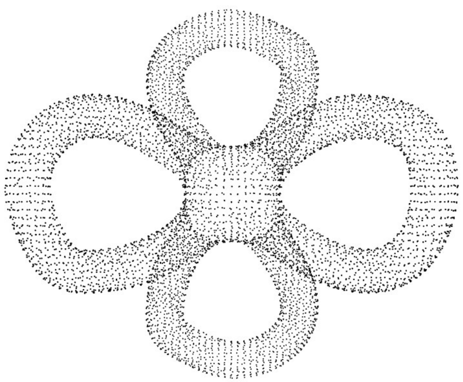
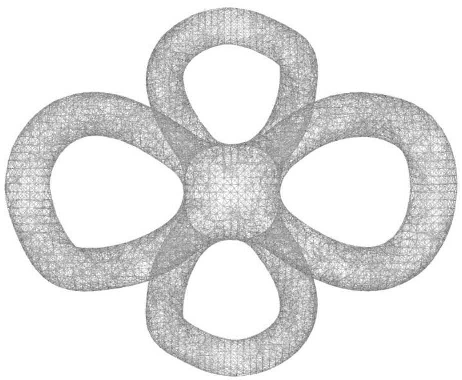
 →  →

 →  →

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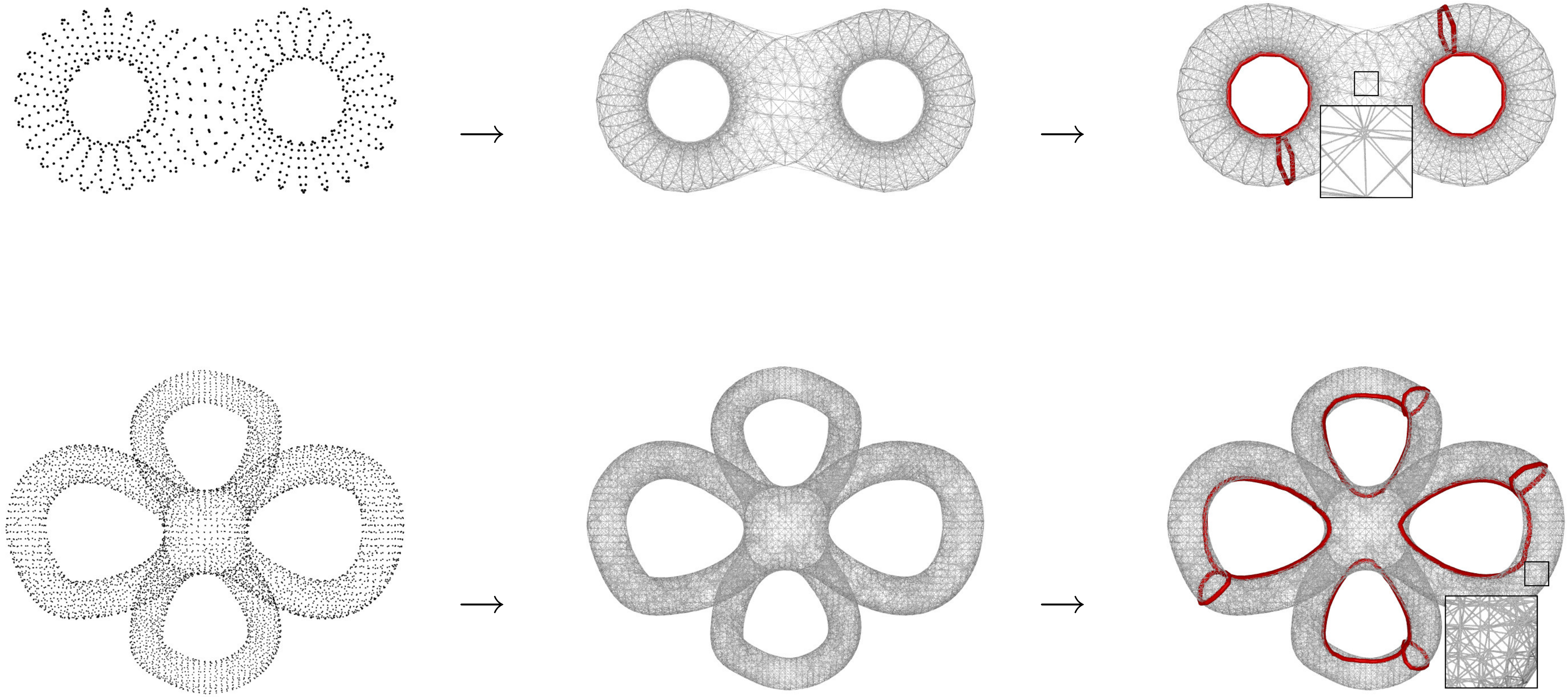

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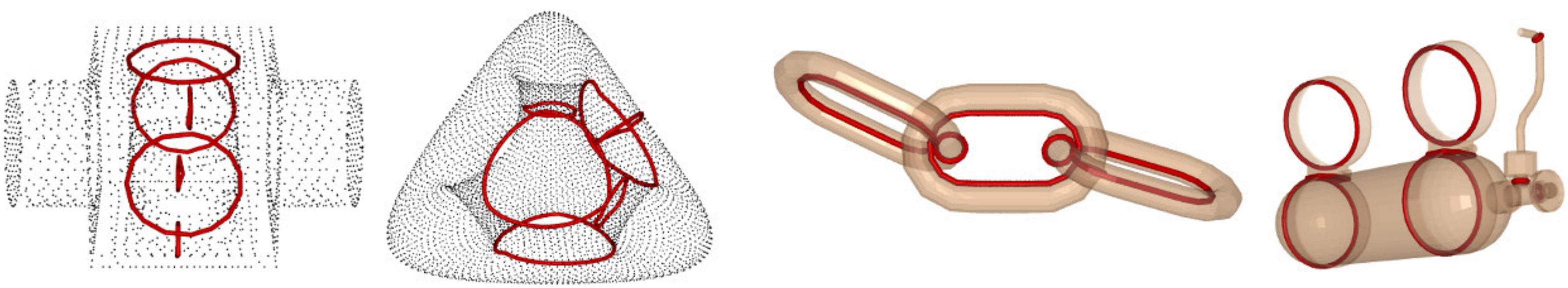

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- 2-manifold triangulations [EW05].
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- A **shortest basis** of $\mathsf{H}_1(\mathbb{T})$ is a set of k loops with minimal length generating $\mathsf{H}_1(\mathbb{T})$.

Theorem 1

- Let $\mathcal{K}$ be a finite simplicial complex with non-negative weights on edges.
- A shortest basis for $\mathrm{H}_1(\mathcal{K})$ can be computed in $O(n^4)$ time where $n = |\mathcal{K}|$.

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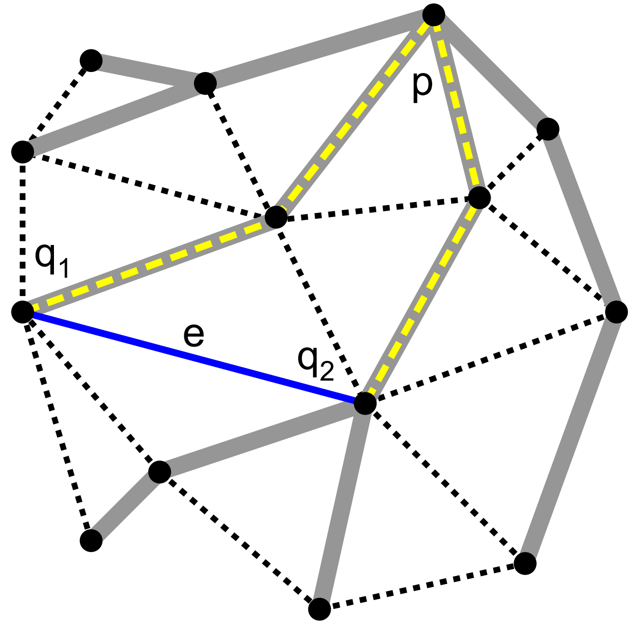
- The **greedy set** G is an *ordered* set of loops $\{g_1, ..., g_k\}$ satisfying the following conditions:
  g_1 is the shortest loop in $\mathcal{L}$ nontrivial in $\mathsf{H}_1(\mathcal{K})$;
  g_{i+1} is the shortest loop in $\mathcal{L}$ independent of $g_1, ..., g_i$.

Canonical loop

- Let T be a shortest path tree in $\mathcal{K}$ rooted at p.
- For $q_1, q_2 \in P$, $\mathrm{sp}_T(q_1, q_2)$ denotes the unique path from q_1 to q_2 through p in T.
- Let E_T be the set of edges in T.
- The **canonical loop** for a non-tree edge e is defined as



$$T(e) = \mathrm{sp}_T(p, q_1) \circ e \circ \mathrm{sp}_T(q_2, p).$$

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- **Proposition** The greedy set chosen from $\cup_{p \in P} G_p$ is a shortest basis of $\mathsf{H}_1(\mathcal{K})$.

Computing G_p

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3: Run the persistence algorithm based on the following filtration of $\mathcal{K}$: vertices in $P = \mathrm{Vert}(\mathcal{K})$, tree edges in T, non-tree edges in the canonical order, triangles in $\mathcal{K}$. Return the set of canonical loops associated with $k = \mathrm{rank}(\mathsf{H}_1(\mathcal{K}))$ edges unpaired after the algorithm.

Computing shortest basis

SPGEN($\mathcal{K}$)

```
For each p ∈ P = Vert(K), set G_p := CANONGEN(p, K).
Sort all loops in ∪_p G_p by lengths in the increasing order.
Let g_1, ..., g_{k|P|} be this sorted list. Initialize G := {g_1}.
for i := 2 to k|P|, do
    if |G| = k, then
        Exit the for loop.
    else if g_i is independent of loops in G, then
        Add g_i to G.
    end if
end for
Return G.
```

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- Whether g is rendeded trivial can be determined by augmenting the filtration of $\mathcal{K}$ with the simplices in $\mathcal{K}'\backslash\mathcal{K}$ and continuing the persistence algorithm.

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- Compute a complex $\mathcal{K}$ from P. Compute a shortest basis of $\mathsf{H}_1(\mathcal{K})$. Argue that if P is *d*ense, a subset of computed loops approximate a shortest basis of $\mathsf{H}_1(M)$ within constant factors.

Complexes

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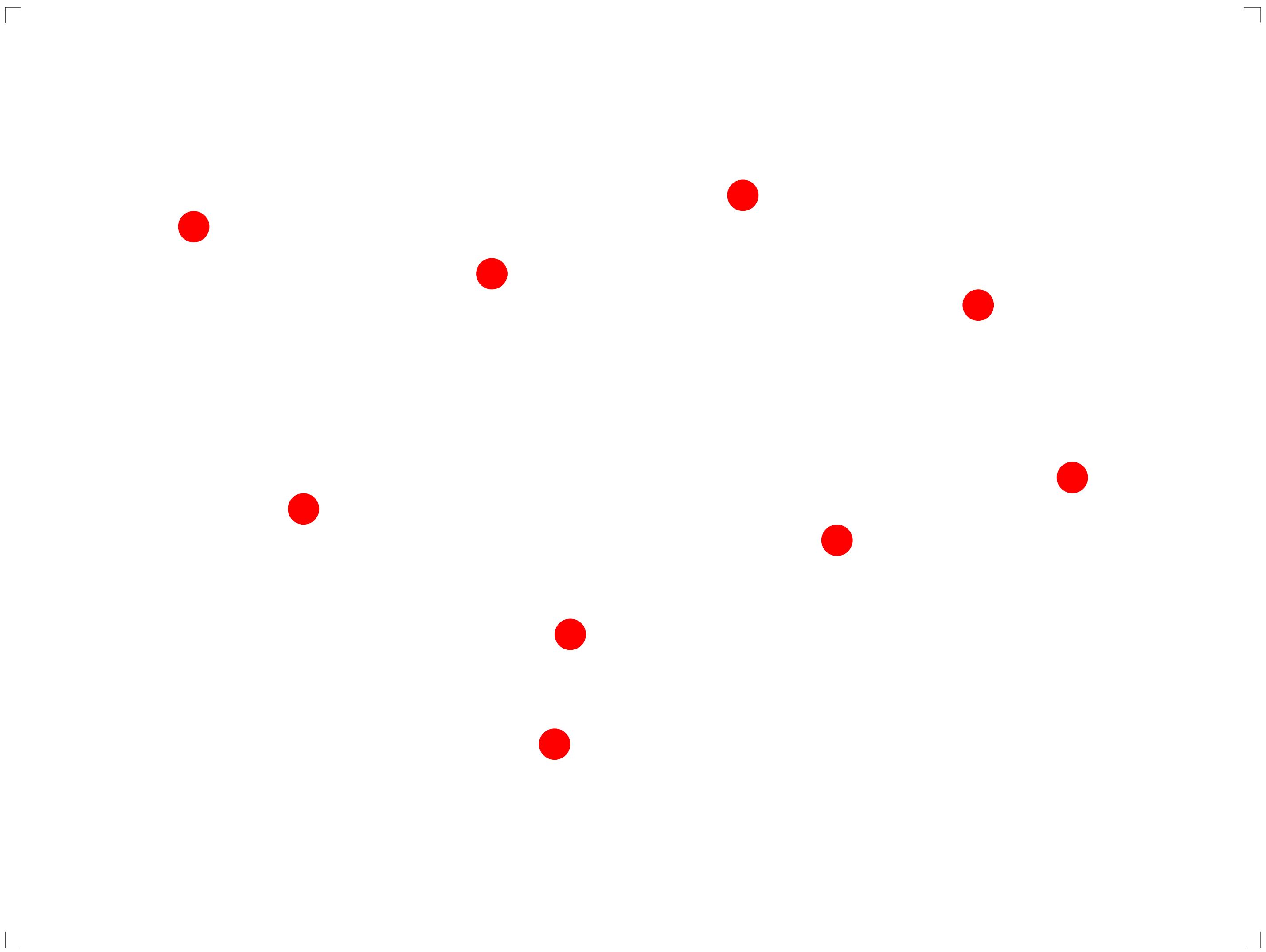
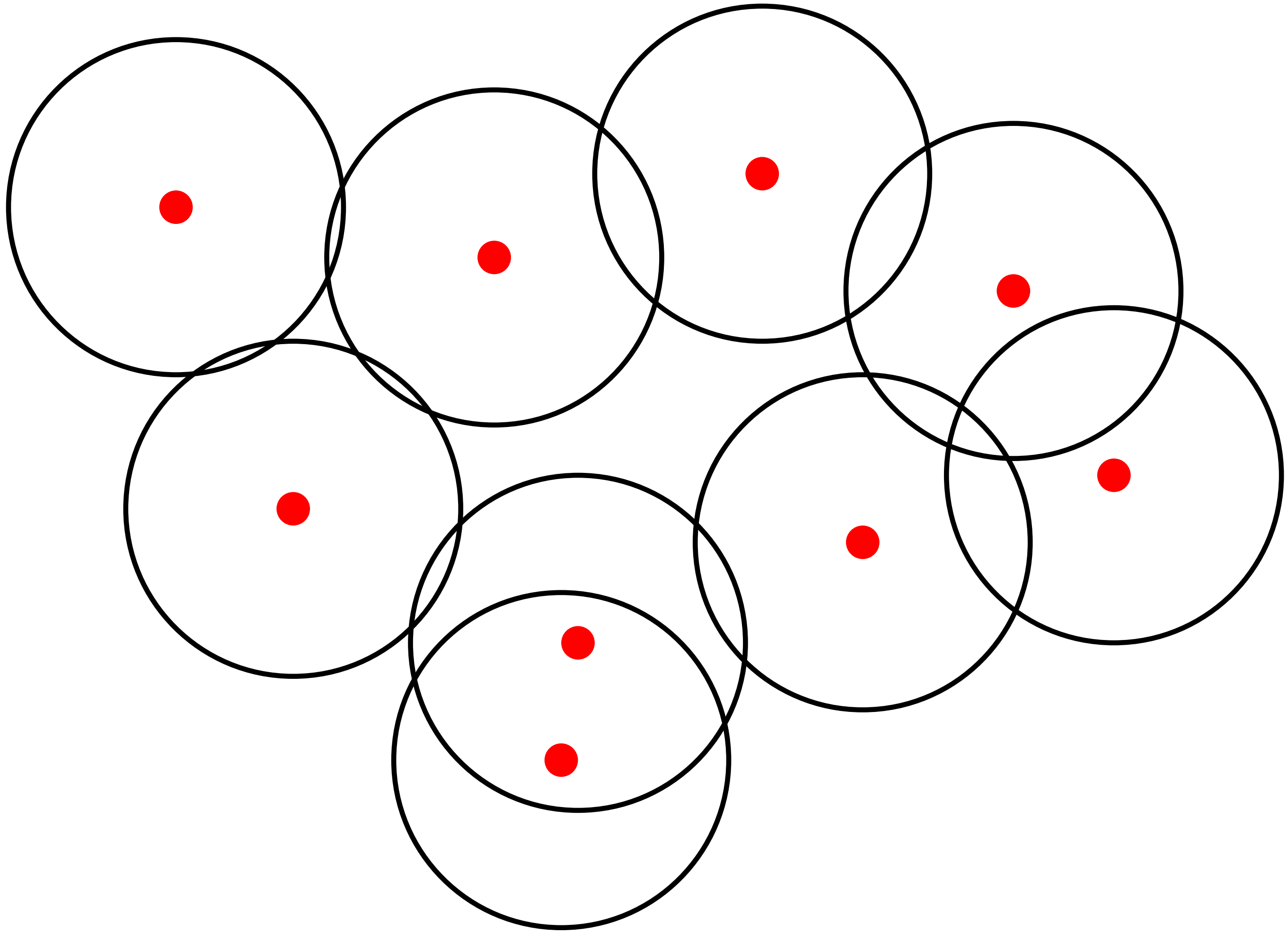
- The **Rips complex** $\mathcal{R}^r(P)$ is a simplicial complex where a simplex $\sigma \in \mathcal{R}^r(P)$ if and only if $\mathrm{Vert}(\sigma)$ are within pairwise Euclidean distance of r.
- **Proposition** For any finite set $P \subset \mathbb{R}^d$ and any $r \geq 0$, one has $\mathcal{C}^r(P) \subseteq \mathcal{R}^r(P) \subseteq \mathcal{C}^{2r}(P)$.

Point set P



Balls $B(p, r/2)$ for $p \in P$



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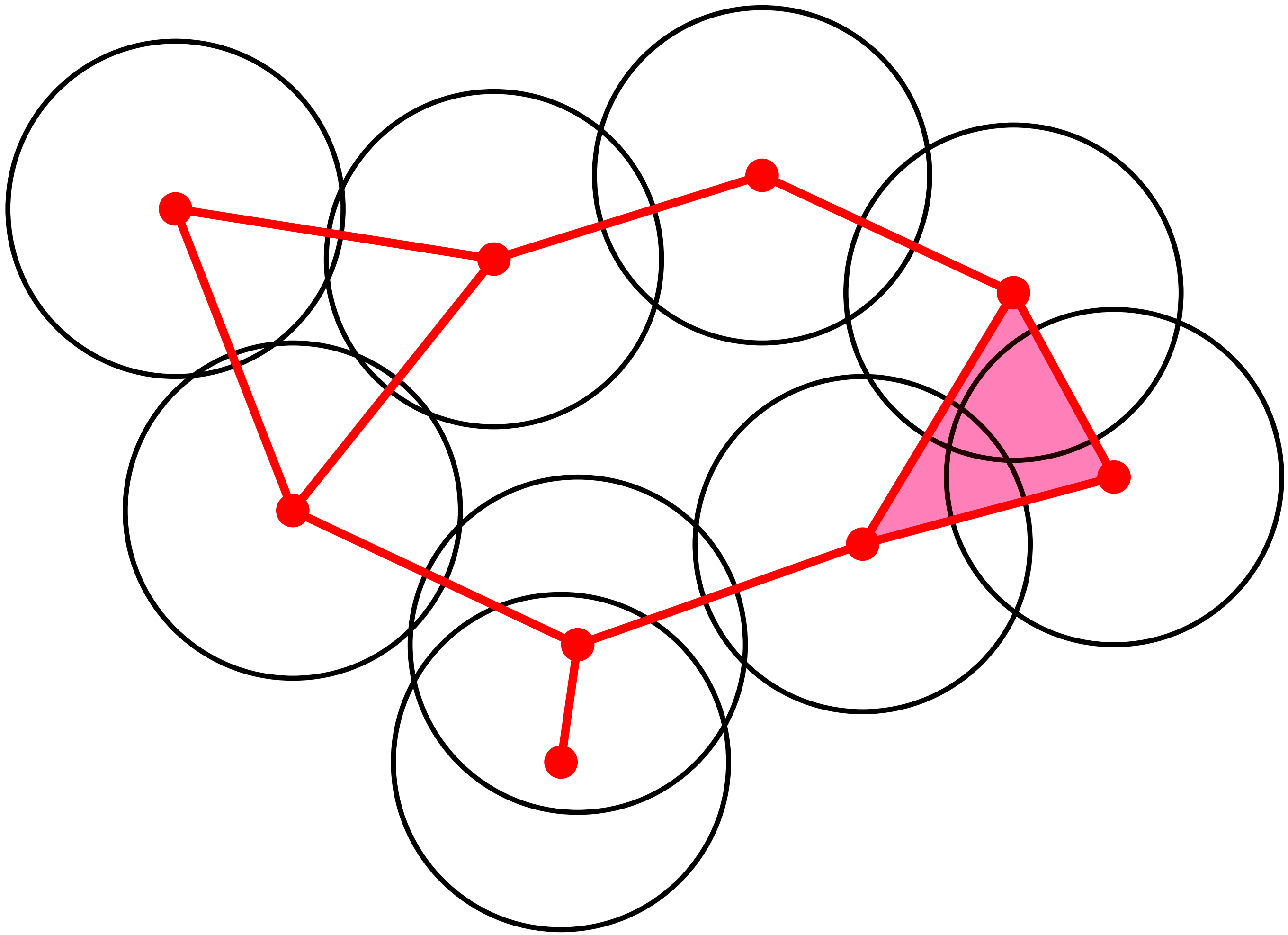
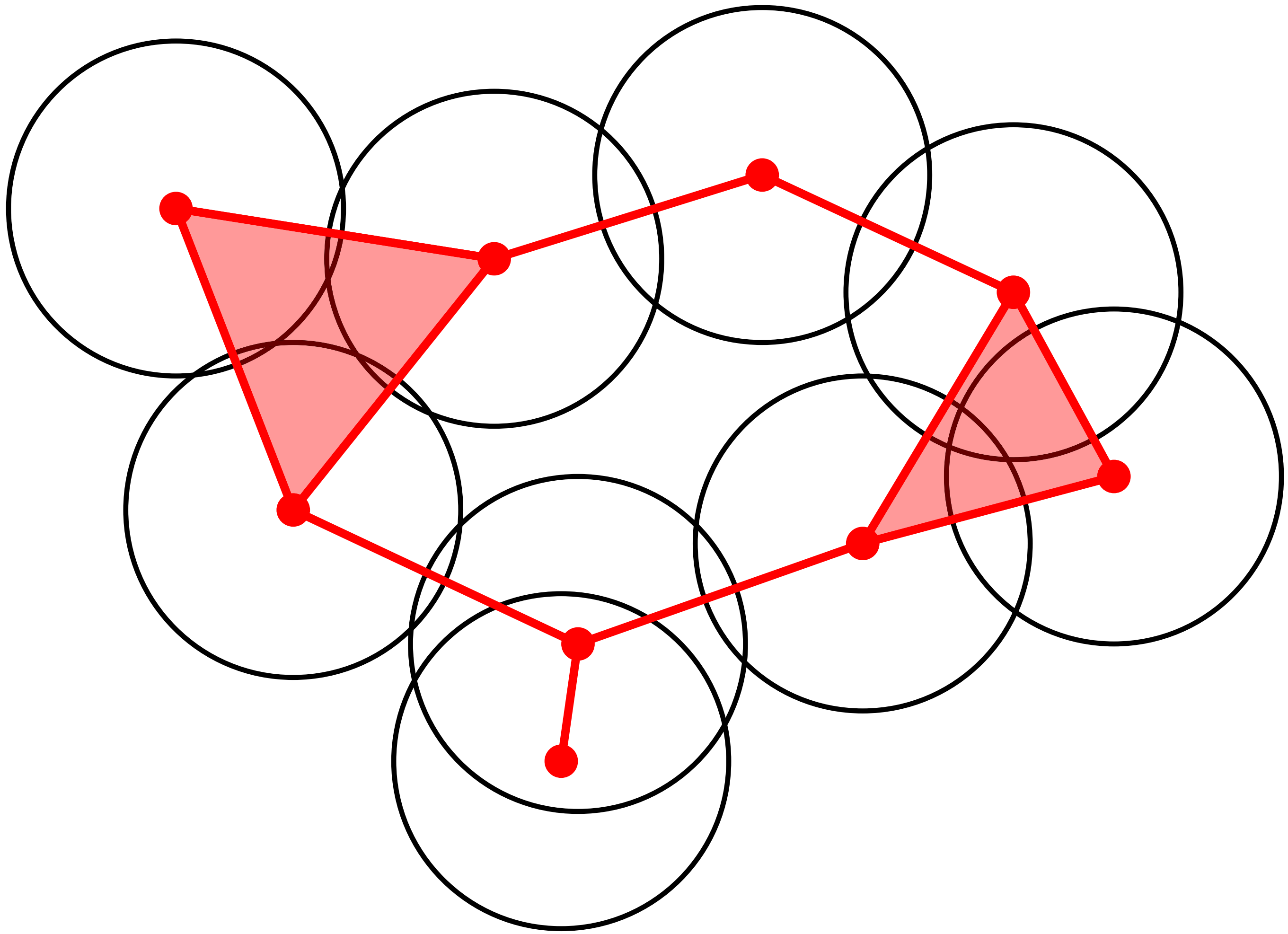


Rips complex $\mathcal{R}^r(P)$



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- P is an ε-sample of M if $B(x, \varepsilon) \cap P \neq \emptyset$ for each $x \in M$.

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- Given a set $P \subset M$ of n points which is an ε-sample of M and $4\varepsilon \leq r \leq \min\{\frac{1}{2}\sqrt{\frac{3}{5}}\rho(M), \rho_c(M)\}$, one can compute a set of loops G in $O(nn_e^2 n_t)$ time where

$$\frac{1}{1+\frac{4r^2}{3\rho^2(M)}} l \leq \mathrm{Len}(G) \leq (1+\frac{4\varepsilon}{r})l.$$

Here n_e, n_t are the number of edges and triangles in $\mathcal{R}^{2r}(P)$.

Using Rips complexes

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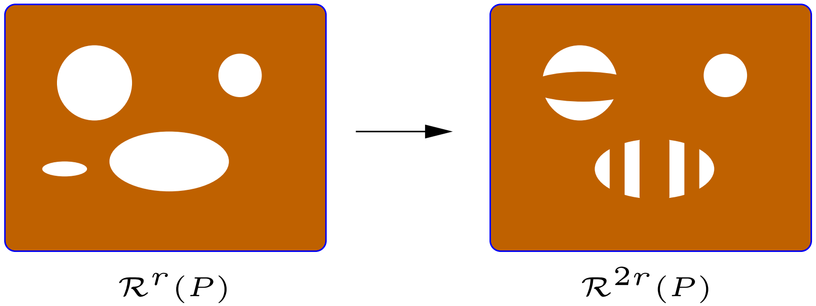
- We weight edges in $\mathcal{R}^{2r}(P)$, creating a complex $\mathcal{K}$. Each edge $e \in \mathcal{R}^{2r}(P) \setminus \mathcal{R}^r(P)$ has a large weight W. Other edges have lengths as their weights.



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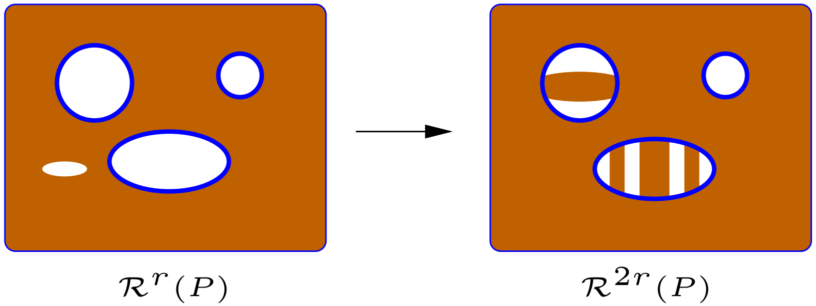

Approximation Algorithm

Theorem SHORTLOOP(P, r) computes a shortest basis for the persistent homology group $\mathsf{H}_1^{r,2r}(\mathcal{R}(P))$.

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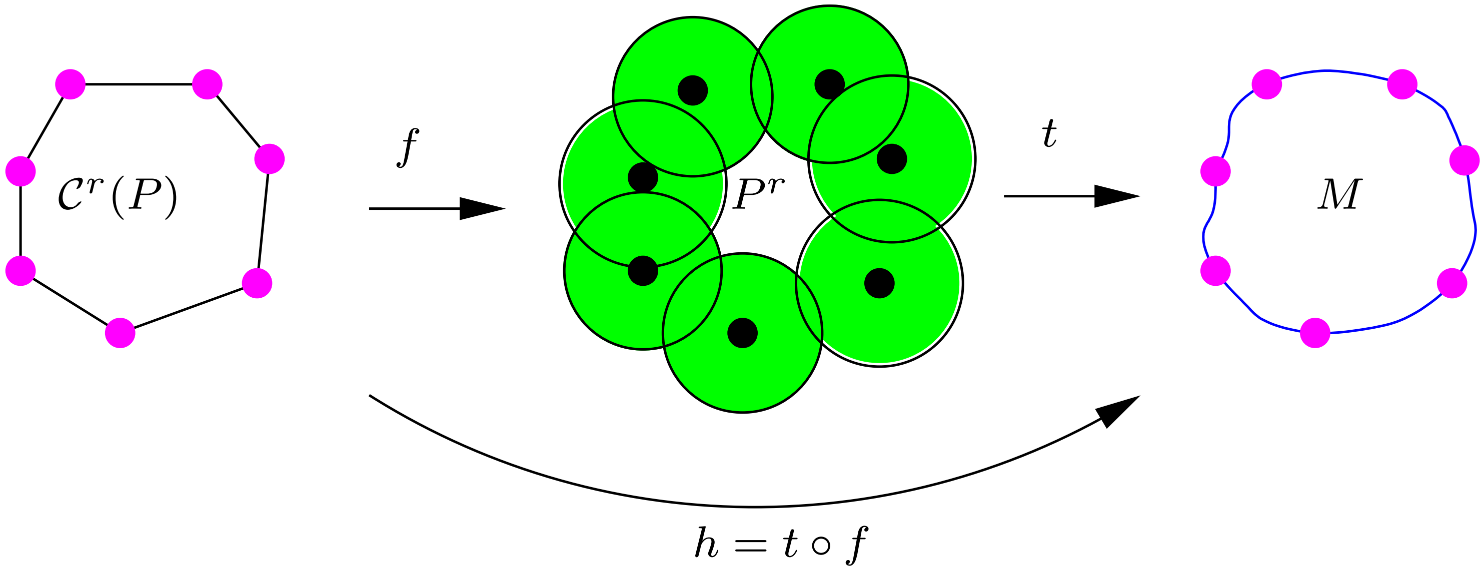
4: Return first k loops from the computed basis where k is the rank of the $\mathsf{H}_1(\mathcal{R}^r(P)) \to \mathsf{H}_1(\mathcal{R}^{2r}(P))$.

Connecting $\mathcal{C}^r(P)$ and M



Bounding Lengths

- Let $P \subset M$ be an ε-sample and $4\varepsilon \leq r \leq \min\{\frac{1}{2}\rho(M), \rho_c(M)\}$.

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- Let g be a geodesic loop in M. There is a loop $\hat{g}$ in $\mathcal{C}^r(P)$ so that $[h(\hat{g})] = [g]$ where h is a homotopy equivalence and $Len(\hat{g}) \leq (1 + \frac{4\varepsilon}{r})Len(g)$.

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- If $G = \{g_1, ..., g_k\}$ and $G' = \{g'_1, ..., g'_k\}$ are the generators of a shortest basis of $\mathsf{H}_1(M)$ and $\mathsf{H}_1(\mathcal{K})$ respectively, then we have $\mathsf{Len}(G') \leq (1 + \frac{4\varepsilon}{r})\mathsf{Len}(G)$.

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- We have Len$(G) \leq (1 + \frac{4r^2}{3\rho^2(M)})Len(G')$.

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- Let G and G' be a shortest basis of $\mathsf{H}_1(M)$ and $\mathsf{H}_1(\mathcal{K})$ respectively.
- We have $\frac{1}{1+\frac{4r^2}{3\rho^2(M)}}\mathsf{Len}(G) \leq \mathsf{Len}(G') \leq (1+\frac{4\varepsilon}{r})\mathsf{Len}(G)$.

Conclusions

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- Software **ShortLoop** is available from authors' web-pages.

Thank you!