Curve Reconstruction:
Connecting Dots with Good Reason

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Abstract

Curve reconstruction algorithms are supposed to reconstruct curves from point samples. Recent papers present algorithms that come with a guarantee: Given a sufficiently dense sample of a closed smooth curve, the algorithms construct the correct polygonal reconstruction. Nothing is claimed about the output of the algorithms, if the input is not a dense sample of a closed smooth curve, e.g., a sample of a curve with endpoints. We present an algorithm that comes with a guarantee for any set P of input points. The algorithm constructs a polygonal reconstruction G and a smooth curve Γ that justifies G as the reconstruction from P.

1 Introduction

Given a set of points sampled on a smooth curve in the plane, the problem of connecting them according to their adjacencies on the curve is the *curve reconstruction* problem. The nontriviality of the problem arises from the absence of any prior knowledge about the curve. The problem, because of its application in computer vision, image processing and pattern recognition has drawn a lot of attention from researchers over the last three decades [15, 16, 17]. If the curve is closed and uniformly sampled, a number of methods is known to work ranging over minimum spanning tree [8], α-shapes

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[3, 7], β-skeleton [11], and r-regular shapes [2]. A survey on these techniques appears in [6]. The case of non-uniformly sampled closed curves was first treated successfully by Amenta, Bern and Eppstein [1] and subsequently improved algorithms such as [5, 9, 10] appeared.

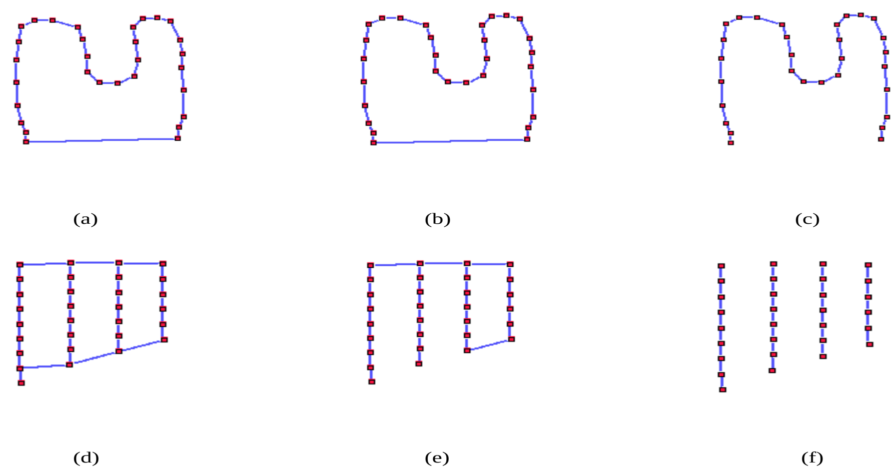

Figure 1: Reconstructions by NN-Crust (a and d), Crust (b and e), and Conservative-Crust (c and f) with $\rho = 2$

We need the following definitions for further exposition. A *single smooth* curve γ is given by a vector function $\mathbf{x}(u) = (x_1(u), x_2(u))$, $u \in U$, where U is the closed unit interval $[0, 1]$ and $\mathbf{x}$ satisfies the property:

i. $\mathbf{x}'(u)$ is continuous and nonzero in the interior of U.

In addition, we require that $\mathbf{x}(u)$ is one-to-one in the interior of U to avoid any self intersection. If $\mathbf{x}(0) = \mathbf{x}(1)$ with $\mathbf{x}'(0) = \mathbf{x}'(1)$, we say that γ is a curve with *endpoints*, and it is a *closed* curve otherwise. A *curve* Γ is a collection of isolated points and single smooth curves that are pairwise disjoint. The *medial axis* of a curve Γ is the closure of all center points of disks *touching* Γ in two or more points. The *local feature size* $f(p)$ at a point p on Γ is the least distance of p from the medial axis. Notice that the property (i) ensures that γ does not have any sharp corner and thus $f(p) \neq 0$ for any $p \in \gamma$. A point set $P \subseteq \Gamma$ is an ϵ*-sample* from Γ if for each point $p \in \Gamma$, its distance to its closest point in P is at most $\epsilon \cdot f(p)$. The *correct reconstruction* of Γ from P is the graph $H = H(\Gamma, P)$ on P such that for each $x, y \in P$, x and y are adjacent in H iff x and y are adjacent on Γ.

Amenta, Bern and Eppstein presented an algorithm Crust that, given an ϵ-sample P from a closed curve for some $\epsilon < 0.252$, computes the correct reconstruction H. Later Dey and Kumar [5] gave an algorithm NN-CRUST that works for $\epsilon < 1/3$. *If P is not an ϵ-sample from a closed curve, no claim is made for either algorithm.*

We present an algorithm CONSERVATIVE-CRUST(P, ρ), where ρ is a non-negative real parameter, that on input P constructs a graph G and a curve Γ. The graph G is a collection of open and closed chains with vertices in P satisfying the following properties:

1. If $\rho < 1/2$ and P is a $(\rho/8)$-sample from a curve Γ' then $H(\Gamma', P) \subseteq G$.
2. There is a positive constant c_0, $c_0 \leq 13.35$, such that for all $\rho < 1/8$, P is a $(c_0\rho)$-sample from Γ and $G = H(\Gamma, P)$.
3. The algorithm can be implemented so that its running time is $O(n \log n)$, where n is the number of sample points.

The first item guarantees that if P is a sufficiently dense sample from a curve (not necessarily closed) then G captures all edges in a correct reconstruction. It may construct additional edges leading us to name the algorithm as CONSERVATIVE-CRUST. Figure 2 suggests that this phenomenon

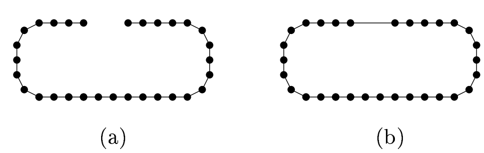

Figure 2: The same point set is a dense sample of a curve with endpoints in (a) as well as a closed curve in (b). A reconstruction algorithm has no possibility to know which.

may be unavoidable. The second item guarantees that the algorithm does not add edges in a haphazard way; it only adds edges for which there is a reason to do so. More precisely, the algorithm provides a curve Γ that justifies G: P is a dense sample from Γ and G is the correct reconstruction of Γ from P. We remark that it is trivial to satisfy item 1 or item 2 independently; the complete graph on P satisfies the first item and the empty graph and $\Gamma = P$ satisfies the second item. The parameter ρ interpolates between these two extreme configurations. As ρ decreases, so does the value of $c_0\rho$ requiring a tighter sampling condition to be satisfied by the output graph G. As a result, G becomes sparser. Thus the range of ρ covers a spectrum of outputs with the complete graph on one end with the large value of ρ, and the empty graph on the other end with ρ being sufficiently small. We experimented with the performance of the algorithm for different values of ρ; see Figure 12 for an example. We see that the range $[1.25, 1.75]$ works well in practice.

Figure 1 shows examples of reconstructions by CRUST, NN-CRUST, and CONSERVATIVE-CRUST from point sets that are not dense samples of a closed curve. Observe, that the former two algorithms include edges in the reconstruction, which one might call "unjustified", and the new algorithm does not.

2 Preliminaries

Our algorithm and the proof of its correctness use the concepts of *Voronoi* and *Delaunay* disks of edges. The vertices of the Voronoi diagram of a point set P are called its *Voronoi vertices.*

Center disk. A *center disk* (C-disk) for an edge e is a disk whose center is at the midpoint of e.

Voronoi disk. A *Voronoi disk* (V-disk) is a disk that is empty of Voronoi vertices of P. A V-disk *for an edge* e is a C-disk for e which is also a V-disk. In Figure 3, the central disk with solid boundary is a V-disk for e.

Delaunay disk. A *Delaunay disk* (D-disk) is a disk that is empty of points in P. A D-disk *for an edge* e is a D-disk with the endpoints of e on its boundary. In Figure 3, the two disks with dashed boundaries are D-disk for the edge e.

Gabriel edge. A *Gabriel edge* is an edge with endpoints in P whose diametric disk is a D-disk. The *Gabriel graph* of P is the graph of all Gabriel edges and is denoted $\mathcal{G}(P)$.

Edge-disk angle. Let e be an edge intersecting the boundary of a disk B in two points (possibly endpoints), p and q. The angle between e and B is the (nonobtuse) angle between e and either of the tangents to B at p and q. In Figure 3, the angle between e and the disk with radius R' is α.

Notations. We will use $\ell(e)$ to denote the length of an edge e, $B(p,r)$ to denote the disk centered at a point p with radius r and $B(e,r)$ to denote the C-disk of e with radius r.

Observation 1 *Let B be a disk of radius R and e be an edge with endpoints on the boundary of B. Then the angle between e and B is $\sin^{-1}(\ell(e)/2R)$.*

The following lemma shows certain properties of V-disks and D-disks.

Lemma 2 *Let e be a Gabriel edge. If $B(e, R)$ is a V-disk, then e has D-disks of radius $R' = \sqrt{R^2 + (\ell/2)^2} > R$ at an angle $\sin^{-1}(\ell/2R') < \sin^{-1}(\ell/2R)$, where $\ell = \ell(e)$.*

Proof. Consider the disks with the endpoints of e on their boundary and

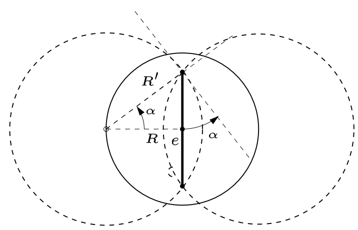

Figure 3: V-disk implies D-disks

with the center on the line bisecting e. Among these disks the ones with the center at a distance at most R from the midpoint of e must be empty of points from P. Otherwise, there would be a Voronoi vertex at a distance less than R from the midpoint of e. See Figure 3. The angle bound follows from the previous observation. ◻

Parts (i) and (ii) of the following lemma can be derived from the previous work of Amenta, Bern and Eppstein [1], and part (iii) from Dey and Kumar [5].

Lemma 3 *Let P be an ϵ-sample from a curve Γ and H be the correct reconstruction $H(\Gamma, P)$. Then:*

(i) For $p, q \in \Gamma$, $f(q) \le f(p) + \ell(pq)$.

(ii) If the intersection of Γ with either B or its relative interior is not simply connected (either it is a closed curve or it consists of at least two disconnected pieces), then B contains a medial axis point.

(iii) Let $\epsilon < 1$. For $e \in H$ and p an endpoint of e, $\ell(e) < \frac{2\epsilon}{1-\epsilon} f(p)$.

3 Algorithm

The algorithm takes as input the point set P and a parameter ρ. It outputs a plane graph G with vertex set P and a curve Γ that witnesses the correctness of the reconstruction.

Algorithm CONSERVATIVE-CRUST(P, ρ)

1. Compute the Delaunay triangulation $\mathcal{D}(P)$.
2. Extract the Gabriel graph $\mathcal{G}(P) \subseteq \mathcal{D}(P)$.
3. Compute the graph $G' \subseteq \mathcal{G}(P)$, where $e \in \mathcal{G}(P)$ is in G' iff $B(e, \ell(e)/\rho)$ is a V-disk.
4. Refine G' into G by eliminating any $e \in G'$ for which $X = B(e, \ell(e)/4\rho) \cap G'$ contains a degree-0 vertex or a degree-1 vertex not connected to e within X. This is repeated until no such edge remains.
5. Output G as defined above and the curve Γ as detailed in Section 6.

The following theorem is the main result of this paper which says that CONSERVATIVE-CRUST constructs the graph $G = G(P)$ with provable reason.

Theorem 4 *G and Γ satisfy the following two properties:*

(i) If $\rho < 1/2$ and P is an $(\rho/8)$-sample from a curve Γ' then $H(\Gamma', P) \subseteq G$.

(ii) Let $c_0 = 13.35$. If $\rho < 1/8$ then P is a $(c_0\rho)$-sample from Γ and $G = H(\Gamma, P)$.

The theorem states that G captures the edges induced by Γ', and that if additional edges are captured in G they are so with a reason: There is a Γ for which G is the correct reconstruction and P is a sample with density no more than a constant times ρ. In our analysis, we obtain the value 13.35 for that constant, but we have not strived to obtain the best bound possible.

The parameter ρ allows to fine-tune the algorithm. As it is made smaller, G becomes sparser. A small value of ρ should be used if the input points are believed to be a very dense sample of the underlying curve.

Although at first sight Steps 3 and 4 of the algorithm seem time consuming, it turns out that they can be implemented without increasing the asymptotic running time of the Delaunay triangulation computation in Step 1.

Theorem 5 CONSERVATIVE-CRUST *can be implemented so that its running time is* $O(n \log n)$.

We prove Theorems 4 and 5 in the next sections.

4 Good Edges are Captured

In this section we prove Theorem 4(i). Let Γ' be a curve and P a $(\rho/8)$-sample from Γ'. We show that $H(\Gamma', P) \subseteq G$ if $\rho < 1/2$. Let $\epsilon = \rho/8$.

Lemma 6 *For each edge* $e \in H(\Gamma', P)$*:*

(i) If $\rho < 8/3$, *e is a Gabriel edge.*

(ii) If $\rho < 4/3$, $B(e, \ell(e)/\rho)$ *is a* V*-disk.*

(iii) If $\rho < 1/2$, *e is not deleted in Step 4.*

Proof. Consider an edge $e \in H(\Gamma', P)$ with endpoints a and b and let $\ell = \ell(e)$. The disk $B = B(a, f(a))$ contains no medial axis point. We show that it would, if either condition fails.

Part (i): If $\rho < 8/3$ and hence $\epsilon < 1/3$, we have $f(a) > \ell(e)(1-\epsilon)/(2\epsilon) > \ell(e)$ by Lemma 3(iii). Thus B contains the diametric disk D of e. If D contained a point in P, this point would not lie between a and b on Γ' (since $e \in H(\Gamma', P)$)i. This can happen only if D intersects Γ' in more than one component. Applying Lemma 3(ii) it follows that D and hence B contains a medial axis point, a contradiction.

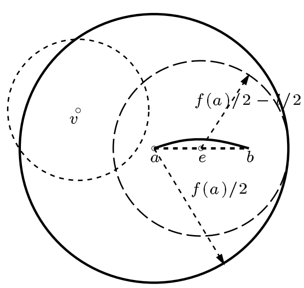

Figure 4: A correct reconstruction edge has a large V-disk

Part (ii): First, we claim that $B(a, f(a)/2)$ is a V-diskand then use it to show $B(e, \ell(e)/\rho)$ is a V-disk. For if there is a Voronoi vertex v in that disk then, for some $r \le f(a)/2$, $B(v,r)$ is a D-disk whose boundary intersects Γ' in at least 3 points (the points defining the Voronoi vertex), and by Lemma 3(ii) there is a medial axis point inside this disk (Figure 4). This medial axis point must reside in $B(a, f(a))$, a contradiction. Now, since $B(a, f(a)/2)$ is a V-disk, so is $B(e,r)$ where $r = f(a)/2 - \ell/2$. Using Lemma 3(iii), we have $r > ((1-\epsilon)/(4\epsilon) - 1/2)\ell(e) = (1-3\epsilon)/(4\epsilon)\ell(e) > \ell(e)/(8\epsilon) = \ell(e)/\rho$.

Part (iii): If such a deletion were possible, let $e \in H(\Gamma', P)$ be the first such edge that gets deleted. Then $X = B(e, \ell/4\rho) \cap G'$ contains a vertex of G' not connected to e within X. This vertex, say v, must be a degree-1 vertex of $H(\Gamma', P)$ since e is assumed to be the first edge deleted in Step 4. Thus $B(e, \ell/4\rho)$ contains at least two points of Γ' that are not connected within X. One of these points is any endpoint of e (for $\rho < 1/2$, $B(e, \ell/4\rho)$ must contain the endpoints of e) and the other point is v. Therefore, $B(e, \ell/4\rho)$ contains a medial axis point by Lemma 3(ii). Since $\ell/(4\rho) + \ell/2 \le \ell(1-\epsilon)/(2\epsilon)$ for $\epsilon \le 15/32$ and since $f(a) > \ell(1-\epsilon)/(2\epsilon)$ according to Lemma 3(iii), we conclude that B does not contain a medial axis point, a contradiction.

$\square$

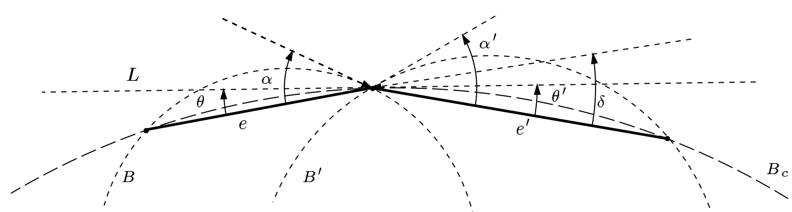

Figure 5: Angle between adjacent edges.

5 The Structure of G

We start with two simple but useful observations. They give lower bounds on the size of the D-balls of the edges of G.

Observation 7 *Each edge $e \in G$ has D-balls of radius $\sqrt{(\ell/\rho)^2 + (\ell/2)^2} > \ell/\rho$, where $\ell = \ell(e)$.*

Proof. This follows from Lemma 2 and the condition in Step 3 of the algorithm. $\square$

Observation 8 *If $\rho \leq 1/2$: Let $e, e' \in G(P)$ with e' intersecting $B(e, \ell(e)/4\rho)$. Then e' has D-disks of radius greater than $\ell(e)/2\rho$.*

Proof. If $\ell(e') > \ell(e)/2$ the claim follows from Observation 7. If $\ell(e') \leq \ell(e)/2 \leq \ell(e)/4\rho$ then e' is contained in $B(e, \ell(e)/2\rho)$. Since $B(e, \ell(e)/\rho)$ is a V-disk we conclude that e' has a V-disk of radius $(1 - 1/2)\ell(e)/\rho$. The claim follows from Lemma 2. ◻

The *turn* from an edge $e = \vec{uv}$ to an adjacent edge $e' = \vec{vw}$ is the signed angle (smaller than π) from $\vec{uv}$ to $\vec{vw}$. If e and e' are not adjacent, the angle between them is the nonobtuse angle between their supporting lines.

Lemma 9 *G satisfies the following:*

(i) If $\rho < 1$: G consists of a collection of chains (some possibly closed) with two adjacent edges making a turn of magnitude at most $2\sin^{-1}(\rho/2)$.

(ii) If $\rho < 1/2$: For $e, e' \in G$ with e' intersecting $B = B(e, \ell(e)/4\rho)$, e' does not intersect the open stripe between the perpendiculars to e through the endpoints of e, it has at least one endpoint in B, and the angle between e and e' is bounded by $\beta_0 = \sin^{-1}(\rho/2) + 2\sin^{-1}(1/4) + \sin^{-1}(\rho)$.

(iii) If $\rho < 1/4$: For $e \in G$, $B(e, \ell(e)/4\rho) \cap G$ consists of a single connected component.

The first two items also hold for the graph G' produced by the frist three steps of our algorithm.

Proof. We prove the three parts in turn. We show that the first two items hold for the graph G' produced by the first three steps of our algorithm and observe that the fourth step does not invalidate them. Thus they also hold for G.

Part (i): Let e and e' be two adjacent edges, B_c be the disk whose boundary passes through the three endpoints of e and e', and L be the line tangent to B_c at the common endpoint of e and e'. The angle δ between the extension of e and e' is equal to the sum of the angles θ and θ' made by L with e and e' respectively. See Figure 5. Note that $\theta \leq \alpha$ where α is the angle formed by e and its maximal D-disk . This is because the radius of B_c is larger than that of B and the assertion follows from Observation 1. Analogously $\theta' \leq \alpha'$. Therefore $\delta \leq \alpha + \alpha'$ and the result follows by Lemma 2 and Observation 7.

To prove that G' is a collection of chains we show that each vertex of G' has degree at most two if $\rho < 1$. Suppose, on the contrary, G' has a vertex

v with degree more than two. There must exist two consecutive edges e and e' around v with a turn at least $\pi/3$. The turn between e and e' is at most $2\sin^{-1}(\rho/2)$ by Part (i). For $\rho < 1$, this is less than $\pi/3$ reaching a contradiction.

Part (ii): Consider the situation in the top part of Fig. 6 in which e is horizontal, B is the V-disk $B(e, R)$ where $R = \ell(e)/4\rho$, B_1 and B_2 are the D-disks of e of radius $\ell(e)/\rho$, and e' intersects B. Since e' has no endpoint in $B_1 \cup B_2$ at least one endpoint of e' is outside the vertical stripe S bounded by the vertical lines through the endpoints of e. We assume w.l.o.g that the left endpoint of e' lies to the left of the left endpoint of e. If the right endpoint of e' lies in S or to the right of S then $\ell(e') \geq \ell(e)$ (since e' has no endpoint in $B_1 \cup B_2$) and hence the D-disks of e' of radius $\ell(e')/\rho$ contain the endpoints of e, a contradiction.

We have now shown that both endpoints of e' lie to the left of the endpoints of e. We assume further that e' has negative slope and consider its maximal D-disk B' *above* it of radius R' (if e' has positive slope, then we consider the D-disk below e'). If e' is vertical (which cannot be as we will see in a short while) we consider the D-disk to its right. We are interested in the angle between e' and the horizontal.

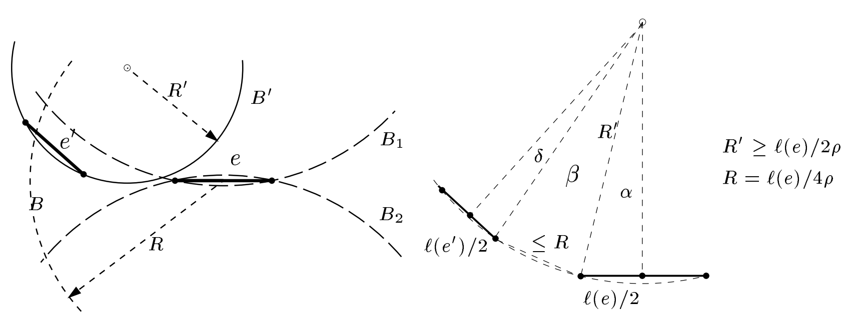

Figure 6: Angle bound.

We have $R' > \ell(e)/2\rho$ by Observation 8. We also have that the right endpoint of e' must lie in B. This follows since e' intersects B, no endpoint of e' lies in $B_1 \cup B_2$, no endpoint of e lies in B', and since the radius of B' is at least twice the radius of B.

Let C' be the boundary of B'. We next transform the situation as follows. We rotate e', together with C', first around the lower endpoint of e' until

C' hits e (this is guaranteed to happen since $R' > \ell(e)/2\rho$ and since the right endpoint of e' lies in B.) and then around the left endpoint of e until both endpoints of e lie on C'. During this process, the angle of e' with the horizontal can only increase. The distance between the left endpoint of e and the right endpoint of e' is not changed by the transformation and is therefore at most R in the final situation.

At the end of the transformation process we are in the situation shown in the lower part of Figure 6. Both endpoints of e lie on C', and the distance between the right endpoints of e' and the left endpoint of e is at most R. The angle between e and e' is $\theta = \alpha+\beta+\delta$ with $\alpha = \sin^{-1}(\ell(e)/2R') \leq \sin^{-1}(\rho)$, $\beta = 2\sin^{-1}(R/2R') \leq 2\sin^{-1}(1/4)$, and $\delta = \sin^{-1}(\ell(e')/2R') \leq \sin^{-1}(\rho/2)$, where the bounds follow from $R = \ell(e)/4\rho$, $R' \geq \ell(e)/2$ and $R' \geq \ell(e')/\rho$.

Part (iii): Let $B = B(e, \ell(e)/4\rho)$ and, for the sake of a contradiction, let G_1 and G_2 be two components of $G \cap B$ of which G_1 contains e. Let $\rho < 1/4$.

First, because of Step 4 of the algorithm, all vertices of G_2 inside B have degree two. Let B_1 and B_2 be the two D-disks of e with radius $\ell(e)/\rho$. Note that $W = B - (B_1 \cup B_2)$ consists of two wedge-like regions W_1 and W_2 disconnected from each other. G_2 cannot connect the two regions W_1 and W_2 by part (ii).

We are left with the only possibility that G_2 enters B along an edge e' and leaves it along another edge e'', without connecting W_1 and W_2; $e' = e''$ is possible. See Figure 7. In this case the turn angles between consecutive edges of G_2 must add to more than $\pi - 2\beta_0$ to accommodate the fact that each of e' and e'' form an angle at most β_0 with e (part (ii)). But, any two consecutive edges cannot make a turn more than $\pi/3$ for $\rho < 1$. This implies that there must be an edge forming an angle more than β_0 with e unless $\pi - 2\beta_0 < \pi/3$, or $\beta_0 > \pi/3$. Clearly, this is impossible if $\rho < 1/4$. ∎

Remark: Since it may appear that Step 4 of our algorithm is needed only for the proof of Lemma 9(iii), we point out that some form of it is necessary, at least for sufficiently small ρ. Consider the configuration of points P in Fig. 8, with $\rho = 1/6$. The corresponding disks in Step 3 for the edges $e = ab$ and $e' = bc$ are B and B'. The edge e' fails the test in Step 3 because of the Voronoi vertex v created by c together with the points x, y (that can be chosen arbitrarily close to each other so that any other Voronoi vertex is quite far). But then, there is a medial axis point in the middle of the omitted edge e' and so, if e is accepted in the reconstruction, P would not be an ϵ-sample for $\epsilon \leq 1/2$.

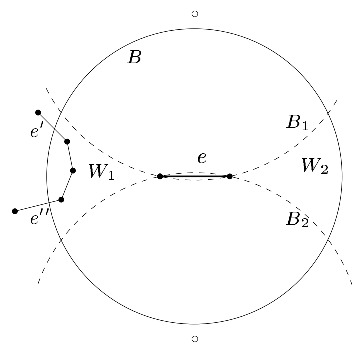

Figure 7: One component.

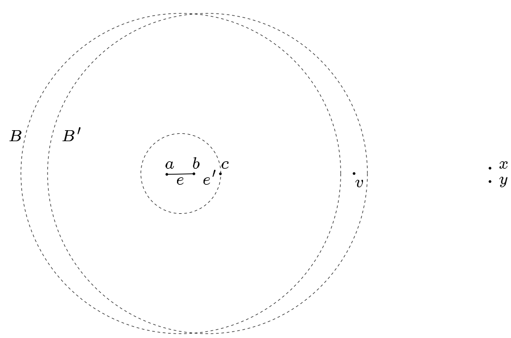

Figure 8: The need for Step 4

6 Curve Fitting

We construct the curve Γ claimed in Theorem 4(ii) edge by edge. We say for any edge e of G that the part of Γ connecting the endpoints of e and containing no other sample is *supported* by e. We denote it with Γ_e.

First, we specify the tangent line to Γ at each point $p \in P$ that has degree 2 in G. The *circumcircle of* p is the circle going through p and its two neighbors. The tangent at p to this circumcircle is our required tangent line. For an edge $e \in G$, Γ_e is defined as follows: If both endpoints of e have degree one, use e itself. If one endpoint of e has degree 1, use the circumcircle of the other endpoint. If both endpoints of e have degree two we proceed as outlined in Figure 9. In this figure e is drawn horizontally and the two neighbors of e turn in opposite directions. The other case is handled similarly. The circumcircle of each endpoint of e determines the tangent at that point. Notice that if e has D-disks of radius R, the radius of each of these circumcircles is at least R.

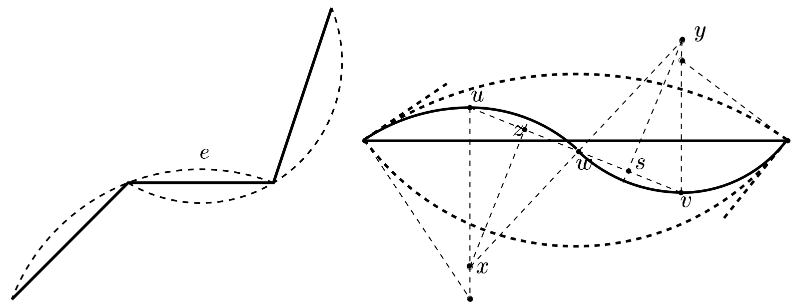

Figure 9: Curve fitting.

Let e correspond to the interval $[0, 1]$. This interval is divided into four equal subintervals, in each of which the curve is a section of a circle satisfying certain constraints. In the first subinterval, the circle section satisfies the tangent constraint at 0 and its tangent is horizontal at $1/4$ (the point u in Figure 9). Thus, the radius of this circle is half the radius of the corresponding circumcircle of the endpoint at 0. Similarly, in the last interval $[3/4, 1]$, the curve is the circle section satisfying tangent constraint at 1 and its tangent is horizontal at $3/4$ (the point v). In the two middle subintervals, two circle sections are determined with the construction illustrated in Figure 9. The point w is the midpoint of uv; xz and ys are perpendicular bisectors of uw and wv respectively. This construction ensures the continuity of the curve and its tangent. We observe that the radii of the circle sections in the middle two subintervals are no less than the smallest of the other two radii. Therefore, if e has D-disks of radius R, the radius of curvature at any point in Γ_e is at least $R/2$. Furthermore, the angle between the edge e and the tangent at any point in this curve section is bounded by the value at the endpoints. This angle is at most $\sin^{-1}(\ell(e)/2R) \leq \sin^{-1}(\rho/2)$ (this is the same bound in Lemma 2 for the angle between an edge and its D-disks).

The constructed curve Γ satisfies the following.

Lemma 10 *Let e be any edge of G. If $\rho < 1/8$ then any point $p \in \Gamma_e$ has a distance at least $\ell(e)(1/10\rho - 1/2) \geq 3\ell(e)/80\rho$ from the medial axis.*

Before we prove this lemma, we show that it implies part (ii) of Theorem 4. Any point on Γ_e has distance at most $\sqrt{(\ell(e)/2)^2 + (\rho\ell(e)/8)^2} = (\ell(e)/2)\sqrt{1 + (\rho/4)^2} \leq 1.001(\ell(e)/2)$ from the closer endpoint of e, since Γ_e

is contained in the intersection of the Delaunay disks of e. Thus, for $\rho < 1/8$, P is an ϵ-sample from Γ for $\epsilon = (1.001\ell(e)/2)/(3\ell(e)/80\rho) = 13.35\rho$.

In the proof of Lemma 10, we will need the following lemma about *minimal* medial axis points. A medial axis point m is called minimal if there is a neighborhood of m such that no point in the neighborhood is the center of a medial axis disk of radius smaller than the medial axis disk centered at m.

Lemma 11 *Let m be a minimal medial axis point and let M be its medial axis disk. Then either M touches Γ in points that are antipodal on M or M touches Γ in a circle section.*

Proof. Assume otherwise, then M touches Γ only in a finite number of points. Let p and q be two touching points that are adjacent on the boundary of M and are not antipodal. Moving m infinitesimally along the angular bisector of the tangents to M at p and q and towards the intersection of the tangents yields a smaller medial axis disk. ◻

We now turn to the proof of Lemma 10.

Proof. (**Lemma 10**) We argue indirectly. Let e be the longest edge of G such that there is a point p on Γ_e whose distance to the medial axis is smaller than $\ell(e)(1/10\rho - 1/2)$. Let m be the medial axis point closest to p and let M be the corresponding medial axis disk. Since M has radius less than $\ell(e)(1/10\rho - 1/2)$, m has distance less than $\ell(e)/10\rho$ from the midpoint x of e and hence M is contained in $B(e, \ell(e)/5\rho)$. The following observation gives information about the features of G touching M.

Observation 12 *Let $B = B(e, \ell(e)/4\rho)$. The medial axis disk M touches Γ at points supported by edges of G having at least one endpoint in B.*

Proof. Observe first that the disk B contains no isolated points of P by Lemma 9(iii) and hence M can touch Γ only at points supported by edges of G. Assume that there is an edge e' such that a point p' on $\Gamma_{e'}$ touches M and such that e' has no endpoint inside B. Then e' does not intersect B, again by Lemma 9(iii). $\Gamma_{e'}$ is contained within the intersection L of its two D-disks with maximum radius. Let $R' \geq \ell(e')/\rho$ be the distance of the midpoint of e' to the centers of the maximal D-disks of e'. Since e' does not intersect B the maximal penetration of L into B is at most $\sqrt{R'^2 + (\ell(e')/2)^2} - R' = R'(\sqrt{1 + (\ell(e')/2R')^2} - 1) \leq R'(\ell(e')/2R')^2/2 \leq \rho\ell(e')/8$.

Since M is contained in $B(e, \ell(e)/5\rho)$ and $\Gamma_{e'}$ touches M, it is necessary that $\rho\ell(e')/8 \geq \ell(e)(1/4\rho - 1/5\rho)$ and hence $\ell(e') \geq 8\ell(e)/20\rho^2 > \ell(e)$. We conclude that the distance from $p' \in \Gamma_{e'}$ to the medial axis is less than $\ell(e)(1/10\rho - 1/2) < \ell(e')(1/10\rho - 1/2)$. This is a contradiction to our choice of e as the longest edge which supports a part of Γ containing a point too close to the medial axis. ◻

Assume that M touches Γ at points $p' \in \Gamma_{e'}$ and $p'' \in \Gamma_{e''}$. If M is not already a minimal medial axis disk, it follows by the argument of Lemma 11 that there is a minimal medial axis disk M' with touching points in Γ between p' and p''. We take $M' = M$ in case M is minimal. By Observation 12 and Lemma 9(iii), we conclude that e' and e'' are in the single component of G within $B(e, \ell(e)/4\rho)$. This implies that the touching points of M' with Γ are also supported by edges in this unique component.

We argue that M' cannot touch Γ in points that are antipodal on M': assume that M' touches Γ at points with tangents t_1 and t_2 and supported by edges e_1 and e_2. Lemma 9(ii) asserts that the angle between e_1 and e_2 is at most $2\beta_0$, and by construction of Γ, the angle between e_i and t_i is at most $\sin^{-1}(\rho/2)$. Since e_1 and e_2 are in the same component, an argument involving the total angle turn between e_1 and e_2 (similar to that in the proof of Lemma 9(iii)) shows that that t_1 and t_2 cannot be parallel. We conclude that M' touches Γ in a circle section and, by Observation 8 and the construction of Γ, that the radius of M' and hence the radius of M is at least $\ell(e)/4\rho$. This contradicts that M is inside $B(e, \ell(e)/5\rho)$. So, there is no edge e violating the statement of the lemma. ◻

7 Algorithm Details

Prime data structures used in the algorithm CONSERVATIVE-CRUST are Delanuay triangulations and Voronoi diagrams both of which can be computed in time $O(n \log n)$ for an n point set P. Step 2 and 3 can be performed analogously.

Step 2: In this step we locate the midpoint of a Delaunay edge e in the Voronoi diagram of P and check if its closest point in P is at a distance less than $\ell(e)/2$.

Step 3: For this step, we locate the midpoint of edges in the Voronoi diagram of the Voronoi vertices of P and check if the closest vertex is at a distance less than $\ell(e)/\rho$, or not. Each point location can be done in time $O(\log n)$ taking a total time $O(n \log n)$.

Step 4: Step 4 of the algorithm could be time consuming since deletion of an edge may trigger another edge deletion in this step and thus can have a cascading effect. However, we show that, still we can carry out this step in time $O(n \log n)$.

We execute deletions in two stages, stage I and stage II. In stage I we detect all edges e whose C-disk B of radius $\ell(e)/4\rho$ contains a degree-0 or degree-1 vertex of G' not connected to e within B. Stage II takes care of those edges that get affected due to deletion of other edges. Use W to denote the degree-0 and degree-1 vertices of G'.

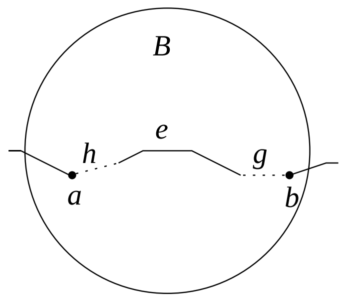

Figure 10: Deletion of edges in Stage II.

Stage I: For stage I we use the order-2 and order-3 Voronoi diagram of W, which can be constructed in time $O(n \log n)$ [14]. We employ them to determine for the midpoint x of every edge e of G' the three closest points in W. The closest point, second closest point and the third closest point to x can be determined from the Voronoi, order-2 Voronoi and the order-3 Voronoi diagram of W respectively. Only two out of these three points can be connected to e within B. We delete e if one of these points is contained in B and is not connected to e within B. The latter condition is tested as follows. A degree-0 vertex is certainly not connected to e and a degree-1 vertex is not connected to e within B if the edge incident to it points away from e. This follows from the angle bound in Lemma 9(ii). Recall that this Lemma applies to the graph G'.

Stage II: Consider the deletion of an edge e in stage II. Let G be the current graph, let T be the chain in G containing e, and let T' be the chain in G' containing e. Let g and h be the edges in $T' \setminus T$ incident to the endpoints of T. Either g or h or both may not exist; if either one exists it was deleted in either step I or in step II before the deletion of e. Let a and b be the endpoints of g and h that do not belong to T. Since e was not deleted in stage I, T' is the only component of G' intersecting $B = B(e, \ell(e)/4\rho)$. If e is to be deleted now there must be a degree-0 or degree-1 vertex v of G and hence T' in B that is not connected to e in $T \cap B$. Presence of v

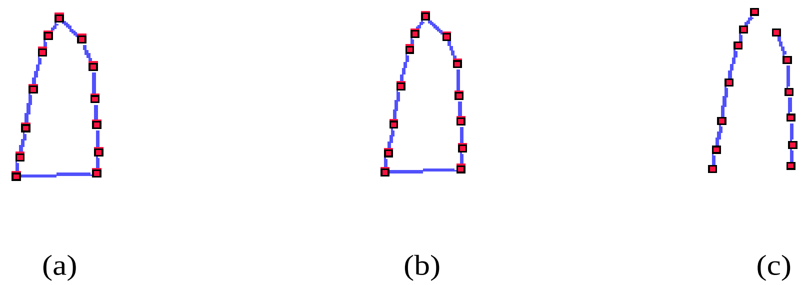

Figure 11: Sharp corner joined by NN-CRUST (a) and CRUST (b) but not by CONSERVATIVE-CRUST (c)

guarantees that either a or b must exist any of which can be taken as v.

The previous paragraph suggests the following recursive procedure for stage II. Let T be any current chain of edges left after deletion of edges in stage I or stage II. The recursive procedure is supplied with two arguments h and g along with T. These two arguments are the two edges that have been deleted already and are adjacent to the two endpoints of T. If both of these arguments are empty, there is nothing to do for T. Otherwise, we start traversing T from both ends *in tandem*, that is, we move the two pointers of the traversals alternatively. During the traversal, when visiting an edge e, we check whether it has to be deleted due to the points called a and b above. If such an edge e is found, we split the chain T into two chains by deleting e and recursively traverse the two chains after providing the appropriate arguments in two recursive calls. The correctness of this procedure follows from the discussion above.

Time analysis: We have already argued that the steps 1, 2 and 3 can be performed in time $O(n \log n)$. Stage I of step 4 takes time $O(n \log n)$ to construct the order-2 and order-3 Voronoi diagrams. Searching in these diagrams takes time $O(\log n)$ per point location summing upto $O(n \log n)$ in total. For stage II, we observe that the time required to split a chain T is proportional to the length of the smaller chain resulting from the split and hence can be charged to the edges of the smaller chain. In this way every edge is charged at most $\log n$ times since every time an edge gets charged, it finds itself in a chain of at least half the length of the previous chain it belonged to. Thus, the total time for stage II is $O(n \log n)$.

8 Experimental Results

We implemented our algorithm using LEDA [12]. We compared the output with those computed by the algorithms CRUST [1] and NN-CRUST [5]. In a number of cases CONSERVATIVE-CRUST could detect the endpoints correctly, where the CRUST and NN-CRUST joined them with edges. Figure 1 shows the output for two such examples.

We experimented with different values of ρ. Figure 12 shows the output for several values of ρ. As expected, for low values of ρ the output contains too few edges, and for large values of ρ the output contains "unnecessary" edges. We found that the range $[1.25, 1.75]$ for ρ works well in practice.

Our algorithm also seems to handle random noise better. Figure 13 shows the output of CRUST and CONSERVATIVE-CRUST where a set of random points is added to the sample. Observe that CONSERVATIVE-CRUST isolates these noise points better than CRUST. One could put a criterion that any point which is not on a component with at least k edges, say $k > 5$, would be deleted. This would delete most of the noise points in case of CONSERVATIVE-CRUST.

Better isolation of spurious noise by CONSERVATIVE-CRUST results from the fact that it is stricter in allowing edges to be output than CRUST for $\rho \in [1.25, 1.75]$. However, due to the same reason CONSERVATIVE-CRUST also detected endpoints near a "sharp corner" as in Figure 11 where CRUST and NN-CRUST joined them as desired.

Acknowledgement.

We thank Herbert Edelsbrunner for suggesting the name CONSERVATIVE-CRUST for the algorithm.

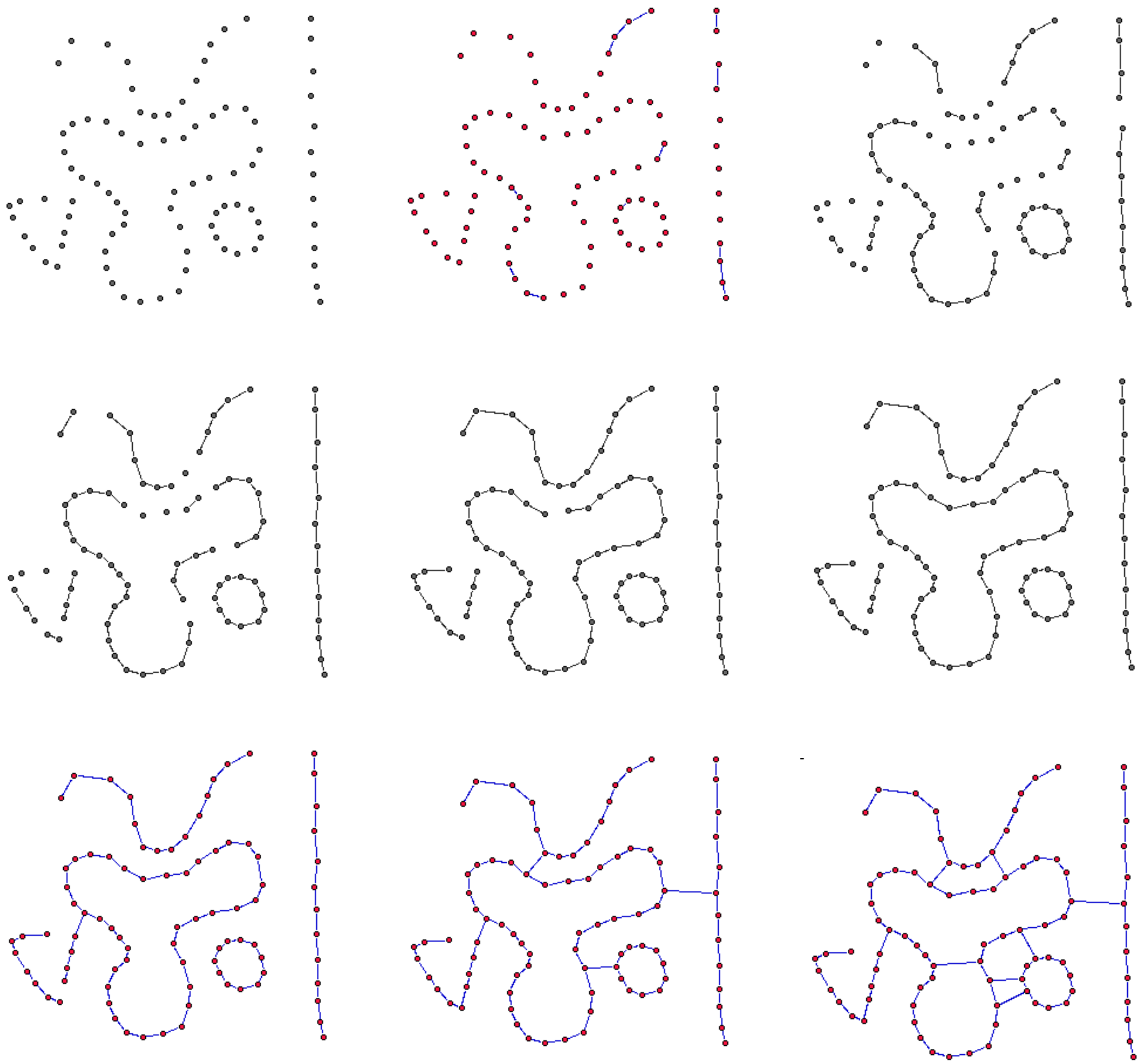

Figure 12: Effect of the parameter ρ: 0, 1/3, 1/2, 3/4, 1, 5/4, 7/4, 2, 2.5.

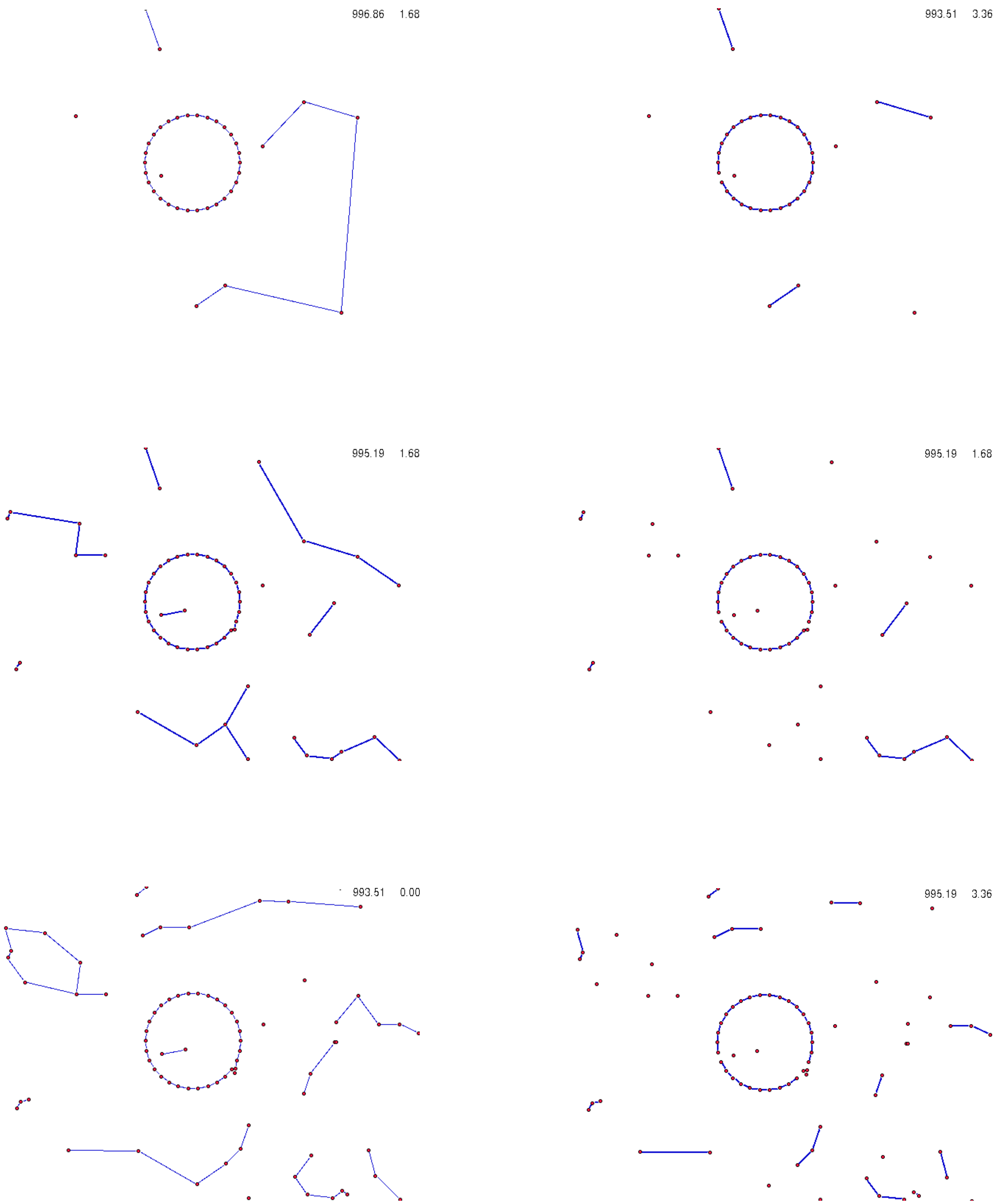

Figure 13: Effect of 10, 30 and 50 noise points on the CRUST (left) and CONSERVATIVE-CRUST (right) algorithms

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