Normal variation for adaptive feature size

Nina Amenta[*] and Tamal K. Dey[†]

September 19, 2007

Background

Let Σ be a closed, smooth surface in $\mathbb{R}^3$. For any two sets $X, Y \subset \mathbb{R}^3$, let $d(X, Y)$ denote the Euclidean distance between X and Y. The local feature size $f(x)$ at a point $x \in \Sigma$ is defined to be the distance $d(x, M)$ where M is the medial axis of Σ. Let n_p denote the unit normal (inward) to Σ at point p. Amenta and Bern in their paper [1] claimed the following:

Claim 1 *Let q and q' be any two points in Σ so that $d(q, q') \leq \varepsilon \min\{f(q), f(q')\}$ for $\varepsilon \leq \frac{1}{3}$. Then, $\angle n_q, n_{q'} \leq \frac{\varepsilon}{1-3\varepsilon}$.*

Unfortunately, the proof of this claim as given in Amenta and Bern [1] is wrong; it also appears in the book by Dey [2]. In this short note, we provide a correct proof with an improved bound of $\frac{\varepsilon}{1-\varepsilon}$.

Theorem 2 *Let q and q' be two points in Σ with $d(q, q') \leq \varepsilon f(q)$ where $\varepsilon \leq \frac{1}{3}$. Then, $\angle n_q, n_{q'} \leq \frac{\varepsilon}{1-\varepsilon}$.*

Definitions and Preliminaries

For any point $p \in \mathbb{R}^3$, let $\tilde{p}$ denote the closest point of p in Σ. When p is a point in Σ, the normal to Σ at p is well defined. We extend this definition to any point $p \in \mathbb{R}^3$. Define the normal n_p at $p \in \mathbb{R}^3 \setminus M$ as the normal to Σ at $\tilde{p}$. Similarly, we extend the definition of local feature size f to $\mathbb{R}^3$. For any point $p \in \mathbb{R}^3$, let $f(p)$ be the distance of p to the medial axis of Σ. Notice that f is 1-Lipschitz. If two points x and y lie on a surface $F \subset \mathbb{R}^3$, let $d_F(x, y)$ denote the *geodesic distance* between x and y. The following facts are well known in differential geometry.

[*]Dept. of CS, U. of California, Davis, CA 95616, amenta@cs.ucdavis.edu
[†]Dept. of CSE, Ohio State U., Columbus, OH 43210, tamaldey@cse.ohio-state.edu

Proposition 3 *Let F be a smooth surface in $\mathbb{R}^3$. Let q and q' be two points in F. Then,*

$$\lim_{d\to 0} \frac{d_F(q,q')}{d(q,q')} = 1.$$

Proposition 4 *Consider the geodesic path between q, q' on a smooth surface F in $\mathbb{R}^3$. Let κ_m be the maximum curvature on this geodesic path. Then $\angle n_q, n_{q'} \leq \kappa_m d_F(q,q')$.*

The Proof

We are to measure $\angle n_q, n_{q'}$ for two points q and q' in Σ. One approach would be to use the propositions above to bound the length of a path from p to q on Σ and then use that length to bound the change in normal direction, but we can get a better bound by considering the direct path from p to q.

Let Σ_ω denote an offset of Σ, that is, each point in Σ_ω has distance ω from Σ. Formally, consider the distance function

$$h\colon \mathbb{R}^3 \to \mathbb{R},\ h(x) \mapsto d(x,\Sigma).$$

Then, $\Sigma_\omega = h^{-1}(\omega)$.

Claim 5 *For $\omega \geq 0$ let p be a point in Σ_ω where $\omega < f(\tilde{p})$. There is an open set $U \subset \mathbb{R}^3$ so that $\sigma_p = \Sigma_\omega \cap U$ is a smooth 2-manifold which can be oriented so that n_x is the normal to σ_p at any $x \in \sigma_p$.*

PROOF. Since $\omega < f(\tilde{p})$, p is not a point on the medial axis. Therefore, the distance function h is smooth at p. One can apply the implicit function theorem to claim that there exists an open set $U \subset \mathbb{R}^3$ where

$$\sigma_p = h^{-1}(\omega) \cap U$$

is a smooth 2-manifold. The unit gradient $(\frac{\nabla h}{\|\nabla h\|})_x = \frac{x-\tilde{x}}{\|x-\tilde{x}\|}$ which is precisely n_x up to orientation is normal to σ_p at $x \in \sigma_p$. ◻

PROOF. [Proof of Theorem 2] Consider parameterizing the segment qq' by the length of qq'. Take two arbitrarily close points $p = p(t)$ and $p' = p(t+\Delta t)$ in qq' for arbitrarily small $\Delta t > 0$. Let $\theta(t) = \angle n_q, n_{p(t)}$ and $\Delta\alpha = \angle n_p, n_{p'}$. Then, $|\theta(t+\Delta t) - \theta(t)| \leq \Delta\alpha$ giving

$$|\theta'(t)| \leq \lim_{\Delta t \to 0} \frac{\Delta\alpha}{\Delta t}.$$

If we show that $\lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t}$ is no more than $\frac{1}{(1-\varepsilon)f(q)}$ we are done since then

$$\begin{aligned}
\angle n_q, n_{q'} &\leq \int_{qq'} |\theta'(t)| dt \\
&\leq \int_{qq'} \frac{1}{(1-\varepsilon)f(q)} dt \\
&= \frac{d(q,q')}{(1-\varepsilon)f(q)} \\
&\leq \frac{\varepsilon}{(1-\varepsilon)}.
\end{aligned}$$

We have $d(q,\tilde{p}) \leq d(q,p) + d(p,\tilde{p})$ and $d(q,p) \leq \varepsilon f(q)$. Since also $\omega = d(p,\tilde{p}) \leq d(p,q) \leq \varepsilon f(q)$, we have $\omega \leq \frac{2\varepsilon}{1-2\varepsilon} f(\tilde{p})$ (by a standard argument using the fact that the function f is 1-Lipshitz). Therefore, $\omega < f(\tilde{p})$ for $\varepsilon < 1/3$, and there is a smooth neighborhood $\sigma_p \subset \Sigma_\omega$ of p satisfying Claim 5.

Let r be the closest point to p' in Σ_ω, and let Δt be small enough so that r and the geodesic between p and r in σ_p lies in σ_p. Notice that , by Claim 5, $\Delta\alpha = \angle n_p, n_{p'} = \angle n_p, n_r$.

Claim 6 $\lim_{\Delta t \to 0} \frac{d(p,r)}{\Delta t} \leq 1$.

PROOF. Consider the triangle prp'. If the tangent plane to σ_p at r separates p and p', the angle $\angle prp'$ is obtuse. It follows that $d(p,r) \leq d(p,p') = \Delta t$. In the other case when the tangent plane to σ_p at r does not separate p and p', the angle $\angle prp'$ is non-obtuse. Let x be the foot of the perpendicular dropped from p on the line of $p'r$. We have $d(p,r)\cos\alpha \leq d(p,p')$ where α is the acute angle $\angle rpx$. Combining the two cases we have $d(p,r)/\Delta t \leq \frac{1}{\cos\alpha}$. Since α goes to 0 as Δt goes to 0, we have $\lim_{\Delta t \to 0} \frac{d(p,r)}{\Delta t} \leq 1$. ◻

Now consider the geodesic between p and r in σ_p, and let m be the point on the geodesic at which the maximum curvature κ_m is realized. Recall that $d_{\sigma_p}(p,r)$ denotes the geodesic distance between p and r on σ_p. Let r_m be the radius of curvature corresponding to κ_m, i.e., $\kappa_m = 1/r_m$. Clearly, $f(m) \leq r_m$. So, Proposition 4 tells us that

$$\Delta\alpha \leq \frac{d_{\sigma_p}(p,r)}{f(m)}.$$

Therefore,

$$\lim_{\Delta t \to 0} \frac{\Delta\alpha}{\Delta t} \leq \lim_{\Delta t \to 0} \frac{d_{\sigma_p}(p,r)}{\Delta t\, f(m)}$$

In the limit when Δt goes to zero, $d_{\sigma_p}(p,r)$ approaches $d(p,r)$ which in turn approaches Δt (Proposition 3 and Claim 6). Meanwhile, $d(q,m) \le d(q,p) + d(p,r)$ approaches $d(q,p) \le \varepsilon f(q)$ as Δt goes to zero (again by Claim 6). So, in the limit, $f(m) > (1-\varepsilon)f(q)$ (again using the fact that f is 1-Lipshitz). Therefore,

$$\lim_{\Delta t \to 0} \frac{\Delta \alpha}{\Delta t} \le \frac{1}{(1-\varepsilon)f(q)}$$

which is what we need to prove. ◻

Remark: The bound on normal variation can be slightly improved to $-\ln(1-\varepsilon)$ by observing the following. We used that $d(q,p) \le \varepsilon f(q)$ to arrive at the bound $f(m) > (1-\varepsilon)f(q)$. In fact, one can observe that $d(q,p) \le \varepsilon t f(q)$ giving $f(m) > (1-\varepsilon t)f(q)$. This gives $|\theta'(t)| \le \frac{1}{(1-\varepsilon t)f(q)}$. We have

$$\angle n_q, n_{q'} \le \int_{qq'} \frac{1}{(1-\varepsilon t)f(q)} dt = -\frac{d(q,q')\ln(1-\varepsilon)}{\varepsilon f(q)} = -\ln(1-\varepsilon).$$

Acknowledgement We thank Siu-Wing Cheng, Jian Sun, and Edgar Ramos for discussions on the normal variation problem.

References

[1] N. Amenta and M. Bern. Surface reconstruction by Voronoi filtering. *Discr. Comput. Geom.* **22** (1999), 481–504.

[2] T. K. Dey. Curve and surface reconstruction : Algorithms with mathematical analysis. Cambridge University Press, New York, 2006.