Improved Bounds on Planar k-sets and k-levels

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Abstract

We prove an $O(nk^{1/3})$ upper bound for planar k-sets. This is the first considerable improvement on this bound after its early solutions approximately twenty seven years ago. Our proof technique also applies to improve the current bounds on the combinatorial complexities of k-levels in arrangements of line segments, k convex polygons in the union of n lines, parametric minimum spanning trees and parametric matroids in general.

1 Introduction

The problem of determining the optimum asymptotic bound on the number of k-sets is one of the most tantalizing open problems in combinatorial geometry. Due to its importance in analyzing geometric algorithms [8, 9, 18], the problem has caught the attention of the computational geometers as well [5, 6, 7, 13, 17, 26, 28]. Given a set P of n points in $\Re^d$, and a positive integer $k \leq n$, a k-set is a subset $P' \subseteq P$ such that $P' = P \cap H$ for a halfspace H and $|P'| = k$. A close to optimal solution for the problem remains elusive even in $\Re^2$. In spite of several attempts, no considerable improvement could be made from its early bound of $O(nk^{1/2})$ given by [19, 24]. Several proofs exist for this well known upper bound [3, 5, 16, 28] which is quite far away from the best known lower bound of $\Omega(n \log k)$ [16]. Pach et al. made the first dent on this upper bound improving it to $O(nk^{1/2}/\log^* k)$. Even such a small improvement in $\Re^2$ was a distinguished result [25]. Recently Agarwal, Aronov and Sharir [3] attacked the problem with a fresh look. Although they could not improve the worst-case upper bound, several approaches were put forward. One of these approaches inspired our proof for the new upper bound of $O(nk^{1/3})$ in $\Re^2$. This is a considerable improvement given the challenging nature of the problem.

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Our proof technique is surprisingly simple. It uses the concept of *crossings* in geometric graphs [1] which was first used by us to prove an $O(n^{8/3})$ bound on 3-dimensional k-sets [13]. Crossings in geometric graphs have been successfully used for many problems in combinatorial geometry. See, for example, [12, 15, 27]. It is expected that our approach would open up new avenues to solve the d-dimensional k-set problem, which remains largely unsolved for $d > 3$ till date. The only nontrivial bound known for $d > 3$ is insignificantly better than the trivial bound [4, 29]. Inspite of this miserable state of the problem, exact asymptotic bound is known for the number of i-sets summed over all $i \leq k$. Alon and Győri [2] showed that this number is $\Theta(nk)$ in $\Re^2$. Clarkson and Shor [11] generalized the bound to $\Theta(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil})$ for $\Re^d$.

Other than k-sets, our proof technique also applies to establish a new $O(nk^{1/3} + n^{2/3}k^{2/3})$ complexity bound for k convex polygons whose edges are non overlapping and lie in the union of n lines. A number of other results follow from this bound. An optimal $\Theta(nk^{1/3})$ bound on the complexity of n-element parametric matroids with rank k follows due to a result of Eppstein [20]. As an immediate consequence, we obtain an $O(EV^{1/3})$ bound on the number of parametric minimum spanning trees of a graph with E edges and V vertices whose edge weights vary linearly with time. A new $O(n^{4/3})$ bound for k-levels in arrangement of n line segments can be derived also from the aforesaid bound.

The paper is organized as follows. We develop major tools for our proof in section 2. A new bound for k-sets in $\Re^2$ is proved in section 3. Section 4 describes the application of our proof technique to other related problems. Finally, we conclude in section 5.

2 Planar k-sets and Line Arrangements

By the well known point-line duality planar k-sets are related to the *levels* in arrangements of lines. A line $\ell : y = ax + b$ is mapped to the point $\ell^* : (a, b)$ and a point $p : (a, b)$ is mapped to the line $p^* : y = -ax + b$ by this duality mapping. An important property of this mapping is that a point p lies above, on, or below a line ℓ if and only if the point ℓ^* lies above, on, or below the line p^* [16]. Let $\mathcal{L}$ be a set of n lines in $\Re^2$ that are dual to a set of n points P in general position in $\Re^2$. This general position assumption is safe since the number of k-sets is maximized for point sets in general positions. Denote the line arrangement of $\mathcal{L}$ by $\mathcal{A}(\mathcal{L})$. For $0 \leq k \leq n-1$, the *k-level* in $\mathcal{A}(\mathcal{L})$ is the closure of all points on the given lines, which have exactly k lines strictly below them. Although the number of k-sets and the number of vertices (complexity) of the k-level are different, they are within a constant factor of each other [16]. The set of vertices of $\mathcal{A}(\mathcal{L})$ with exactly k lines below them is denoted S_k. Each vertex $v \in S_k$ is mapped to a dual line v^* that supports a *k-set edge* e_{pq} passing through two points p, q in P and has exactly k points below it. It is known that the set of vertices in the k-level of $\mathcal{A}(\mathcal{L})$ is $S_k \cup S_{k-1}$. We assume $S_{-1} = S_{n-1} = \emptyset$. See figure 2.1 for an illustration. As observed in [3], at each vertex $v \in S_{k-1}$ the k-level makes a left turn while at each vertex $v \in S_k$ it makes a right turn.

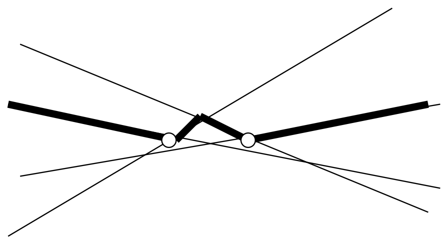

Figure 2.1: Second level in an arrangement of four lines, empty circles denote the vertices of S_1.

2.1 Concave Chains

The most useful idea that we borrow from [3] is the concept of *concave chains*. We define a set of k concave chains $c_1, c_2, ..., c_k$ that are vertex disjoint and cover all vertices of S_{k-1} as follows. The chain c_i starts at $x = -\infty$ and moves along the line with the i-th largest slope. It is observed that the chain reaches the k-level only at a vertex in S_{k-1}. At any vertex $v \in S_{k-1}$ reached by c_i from left, one moves to the right along the other line incident on v. This construction implies that the two edges incident on c_i at v do not belong to the k-level. As a result, the chain c_i makes only right turn at any vertex of S_{k-1}, and thus it is concave. Several properties of these concave chains have been observed in [3]. Perhaps it is appropriate to list some of these important properties here.

i All vertices of S_{k-1} are covered by the concave chain vertices and vice versa.

ii Concave chains are vertex disjoint and non-overlapping.

iii They cover the entire arrangement below the k-level. As a result, all vertices of $\mathcal{A}(\mathcal{L})$ below the k-level appear as crossings between concave chains.

iv All vertices of a chain c_i belong to the lower envelope of the lines contributing to c_i. See figure 2.2 for an illustration.

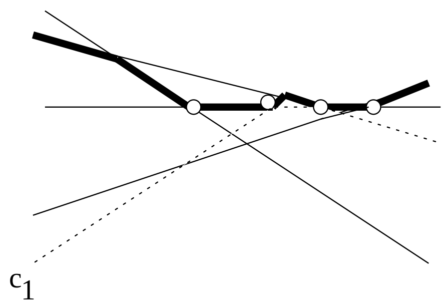

Figure 2.2: The concave chain c_1 corresponding to the third level is shown with dotted line segments.

3 New Upper Bounds in $\Re^2$

3.1 Crossings in Concave Chains

The concave chains as defined in the previous section cross each other below the k-level. The result of [2] implies that the number of vertices below the k-level is at most $O(nk)$. We desire to relate the crossings of the chains with the intersections of pairs of $(k-1)$-set edges in the dual setting. Define a graph $G = (V, E)$ where V is the set of points dual to the lines in $\mathcal{L}$ and E is the set of $(k-1)$-set edges whose supporting lines have exactly $k-1$ points below them. Let e_{pq} and e_{rs} be two vertex disjoint edges in E that cross. Let us examine what does this crossing imply in terms of the chains that we defined. A *double wedge* formed at a vertex v by two lines ℓ_1, ℓ_2 meeting at v is the closed region between the upper and lower envelopes of $\mathcal{A}(\ell_1 \cup \ell_2)$. The edge e_{pq} is mapped to the double wedge formed at the vertex u

where the lines p^* and q^* meet. Similarly e_{rs} is mapped to the double wedge at the vertex v where r^* and s^* meet. The condition that e_{pq} and e_{rs} cross translates to the condition that the line passing through u and v lie in the double wedges of both vertices. This means that the edge connecting u and v is tangent to both chains that contain u and v. Notice that u and v cannot appear on the same chain since otherwise the line passing through u, v must be dual to a common endpoint of e_{pq} and e_{rs}, which does not exist. See figure 3.3. These observations are crucial for our counting the crossings between $(k-1)$-set edges of G. A *common tangent* between two chains c_i and c_j is a line segment that connects a vertex of c_i with a vertex of c_j, and whose supporting line is a tangent to both chains. Let x_{ij} denote the number of crossings between the chains c_i and c_j.

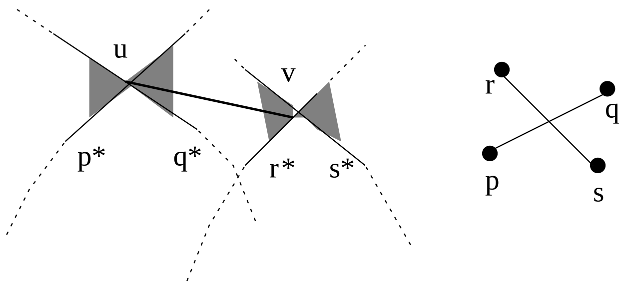

Figure 3.3: A pair of crossing $(k-1)$-set edges and their duals.

LEMMA 3.1 The number of common tangents between any two chains c_i and c_j is at most x_{ij}.

PROOF. All common tangents between c_i and c_j appear on the upper hull of the vertices of c_i and c_j together. Otherwise, the condition that each chain is the lower envelope of the lines supporting its edges is violated. Consider a common tangent T connecting a vertex a on c_i with a vertex b on c_j. Due to our construction it must be true that c_j lies below a and c_i lies below b. However, this swap in vertical ordering cannot be effected without a crossing between c_i and c_j below T since the x-span of each concave chain ranges from $x = -\infty$ to $x = +\infty$. We charge this crossing for T. Since the x-spans of all common tangents are inetrior-wise disjoint, any crossing between c_i and c_j is charged at most once. See figure 3.4 for an illustration. Therefore, the number of common tangents between c_i and c_j is at most x_{ij}. ☐

LEMMA 3.2 Total number of crossings among the edges of G is at most $O(nk)$.

PROOF. From our pervious discussions, it is clear that the total number of crossings among the edges in G is

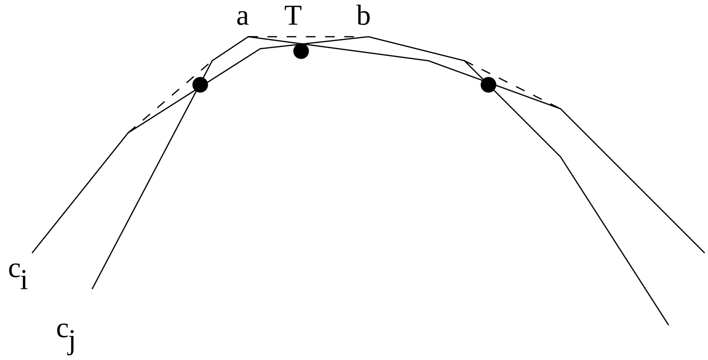

Figure 3.4: Common tangents between two chains are shown with dotted lines. Crossings between the two chains are marked with dark circles.

less than or equal to the total number of common tangents among all possible pairs of chains. This number is no more than the total number of crossings among the chains due to lemma 3.1. The concave chains intersect only at vertices that belong to levels $< k$. Total number of such vertices is at most $O(nk)$ due to a result of Alon and Győri [2]. The claim follows immediately. ☐

3.2 Improved Bound

THEOREM 3.3 There are $O(nk^{1/3})$ k-sets in a set of n points in $\Re^2$.

PROOF. We show that $|S_{k-1}| = O(nk^{1/3})$ for any $1 \leq k \leq n-1$. Since the k-level vertices are in $S_{k-1} \cup S_k$, and the number of vertices in the k-level is within a constant factor of the number of k-sets in the dual, the result follows. In the graph G we have exactly $|S_{k-1}|$ edges. We can assume $|S_{k-1}| > 4n$. Otherwise, the claim follows trivially. It follows from the result of Ajtai, Chvátal, Newborn, Szemerédi [1] and Leighton [23] that there are at least $c \cdot t^3/n^2$ crossings among $t > 4n$ edges connecting pairs of points in a set of n points in plane, where c is some constant. Plugging $t = |S_{k-1}|$ and using the result of lemma 3.2, we obtain $|S_{k-1}|^3/n^2 < c_1 \cdot nk$ for some constant c_1. This immediately provides $|S_{k-1}| = O(nk^{1/3})$. ☐

4 Related Problems

4.1 Convex Polygons and Matroid Optimization

Consider a set of k convex polygons whose edges are non overlapping and are drawn from n lines. The complexity of these polygons is the total number of vertices they have altogether. If they are interior-wise disjoint, an optimal $\Theta(n^{2/3}k^{2/3}+n)$ bound is known [10, 22]. However, these analysis techniques fail if the polygons overlap in their interiors. Our proof technique can be used to establish an optimal $\Theta(nk^{1/3}+n^{2/3}k^{2/3})$ bound for this case.

First, we split each convex polygon into an upper chain and a lower chain. Upper chain consists of all points of the boundary that do not have any point of the polygon strictly above it. Similarly define the lower chains. Without loss of generality we carry out the analysis for the upper chains only. Also, for technical reasons we eliminate chains with only one edge, which have $O(k)$ complexity altogether. The leftmost and the rightmost edges of all upper chains are extended along their supporting lines to $x=-\infty$ and $x=+\infty$ respectively. With this modification, upper chains remain non-overlapping except possibly at their leftmost and rightmost infinite edges. Lemma 3.1 is still valid for these upper chains whose vertices lie on the lower envelope of the supporting lines of their edges. However, while counting over all pairs of upper chains, a vertex v may be charged for all pairs of chains whose infinite edges cross at v. But, these charges accumulated over all pairs are at most $O(k^2)$ since there are $O(k^2)$ pairs of chains, each issuing $O(1)$ such charge. All other crossings between upper chains are charged only once over all pairs. Since any line can cross upper convex chain in at most two points, and any such crossing is counted twice for each participating line, we have at most $2nk/2 = nk$ crossings among k convex upper chains. Thus we collect $O(k^2+nk)$ charges in total, which is an upper bound on the number of common tangents among pairs of upper chains. Using this in combination with the technique of the previous section, we obtain the desired $O(nk^{1/3}+n^{2/3}k^{2/3})$ complexity bound for convex chains. For this we dualize the supporting n lines to n points and consider a graph G as follows. The vertex set of G consists of dual points. Two vertices in G is connected with an edge if and only if the intersection point of their dual lines contributes a vertex on the upper chains. This graph has at most $O(k^2+nk)$ crossings according to our previous arguments. Using the lower bound result on crossings [1], we immediately get the desired bound. This bound is tight since for $k<n$, the first term $nk^{1/3}$ dominates and a matching lower bound is proved in [20]. For $k>n$, the second term $n^{2/3}k^{2/3}$ dominates and a matching lower bound is established by the many-faces result of [10].

THEOREM 4.1 A set of k convex polygons whose edges are non overlapping and lie in the union of n lines have $\Theta(nk^{1/3}+n^{2/3}k^{2/3})$ edges.

In [20] Eppstein showed that an upperbound on the complexity of the class of aforesaid convex polygons also provides an upperbound on the complexity of general parametric matroid optimization problems. He showed an $\Omega(nk^{1/3})$ lower bound for the general n-element parametric matroid optimization problem with rank k. Theorem 4.1 establishes a tight upper bound for it. An immediate implication of this result is the case of parametric minimum spanning trees of a graph with V vertices and E edges where the edge weights vary linearly with time. The previous $O(EV^{1/2})$ bound of Gusfield [21] is improved to $O(EV^{1/3})$ by our result.

4.2 Complexity of j consecutive levels

Let $L_k, L_{k-1}, ..., L_{k-j+1}$ be $j>0$ consecutive levels in an arrangement of n lines. We are interested in determining the complexity of these j levels altogether. By duality this complexity is within a constant factor of the total number of ℓ-sets in the set of dual points where $k \le \ell \le (k-j+1)$. Consider the convex chains partitioning E_ℓ for each ℓ where $(k-j+1) < \ell \le k$. We are interested in counting the number of vertices of all concave chains for all the j levels. In the dual we consider the graph G containing all ℓ-set edges where $k-j \le \ell \le k-1$. The number of crossings among the edges of G is again determined by the number of common tangents among all pairs of concave chains. A common tangent between two chains c_1, c_2 is charged to the vertex v where c_1 and c_2 cross below the common tangent. The vertex v necessarily lie below L_k. Observe that v may be charged for more than one pair of concave chains. Suppose another pair c_2, c_3 charge v. The argument of lemma 3.1 applies to deduce that two pairs of chains coming from the same pair of levels cannot charge v. Hence, c_1, c_2 and c_2, c_3 necessarily come from different pairs of levels. This implies that a vertex is charged at most $O(j^2)$ units, which gives an $O(nkj^2)$ bound on the number of common tangents. Now applying the crossing result of [1] on G we obtain the inequality $t^3/n^2 < c.nj^2k$ for some appropriate constant $c>0$. This immediately gives $t = O(nk^{1/3}j^{2/3})$.

THEOREM 4.2 There are at most $O(nk^{1/3}j^{2/3})$ ℓ-sets summed over $(k-j+1) \le \ell \le k$.

4.3 Line Segments

The question of the k-level complexity can be asked for other geometric structures such as line segments, curves [26]. We extend our technique to line segment arrangements and improve the current bound on their k-levels. This, in turn, implies an improvement of the current bound on the k-levels of arrangements of triangles in $\Re^3$ due to a result of [3].

Let $\mathcal{R}$ be a set of n line segments in $\Re^2$ and $\mathcal{A}(\mathcal{R})$ denote the corresponding arrangement. The k-level in $\mathcal{A}(\mathcal{R})$ is defined as the closure of all points on line segments, which have exactly k line segments strictly below them. Notice that the k-level in this case may have discontinuities. These discontinuities are caused by the endpoints of the line segments where the k-level jumps vertically up or down. See figure 4.5 for an illustration. It is easily observed that the number of such discontinuities is at most $2n$. Hence the complexity of the k-level is dominated by the number of vertices where interiors of two segments meet on the k-level. It is proved in [3] that this number is $O(n^{3/2})$ and thus obtaining an $O(n^{3/2})$ upper bound on the complexity of the k-level. We can employ the technique of previous section to improve this bound to $O(n^{4/3})$. First, we need to extend

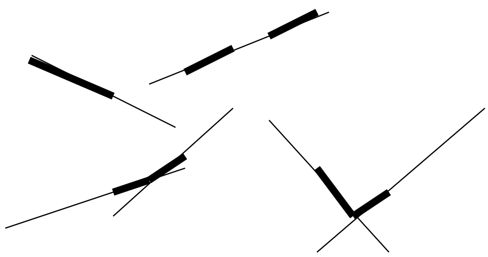

Figure 4.5: First level in a line segment arrangement.

the concept of concave chains to line segment arrangements. This is already done in [3]. We define S_k and the concave chains as before except that the chains start at a left endpoint of a line segment or a discontinuity, and terminates at the right endpoint of some other segment or a discontinuity. There are at most $2n$ such concave chains. Applying the result of section 4.1 with $k = 2n$, we obtain the following theorem.

Theorem 4.3 The complexity of the k-level in an arrangement of n line segments is $O(n^{4/3})$.

The new bound on k-levels in line segment arrangement gives an improved bound on the k-levels in arrangement of triangles in $\Re^3$. This follows from a result of [3]. Plugging in our new bound into the analysis of [3], an $O(n^{25/9})$ bound can be established on the complexity of the k-levels in arrangements of triangles in $\Re^3$.

5 Conclusions

In this paper we provide a considerable improvement of the upper bound of planar k-sets which has defied all such attempts so far except for a small improvement of [25] by a factor of $\log^* k$. The technique is further employed to improve the current best bounds of several other related problems. It remains to be seen if the technique can be used in higher dimensions, albeit with necessary modifications. The generalization of the result of [1] exists [12, 14]. However, the concept of convex chains do not generalize in higher dimensions in a straightforward manner. The author believes that the technique developed in this paper would make further inroads into the challenge of k-set problem and probably into other related combinatorial problems.

Acknowledgements

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