Computing Homology Cycles with Certified Geometry

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The Ohio State University

Collaborators
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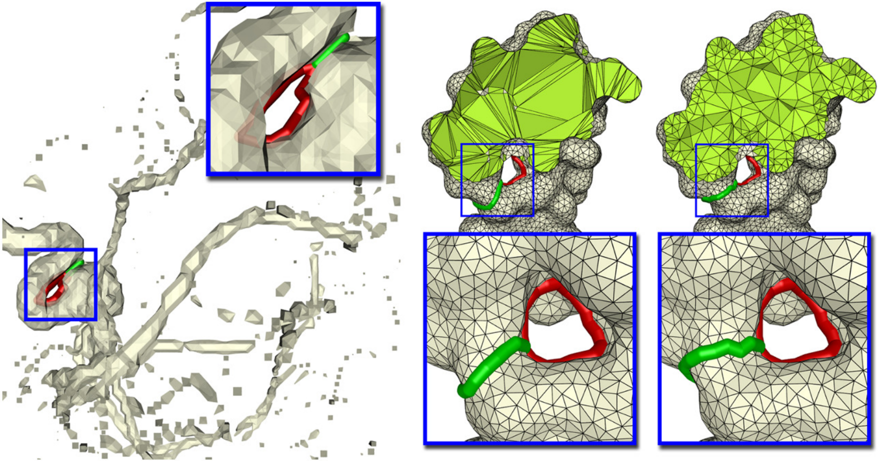
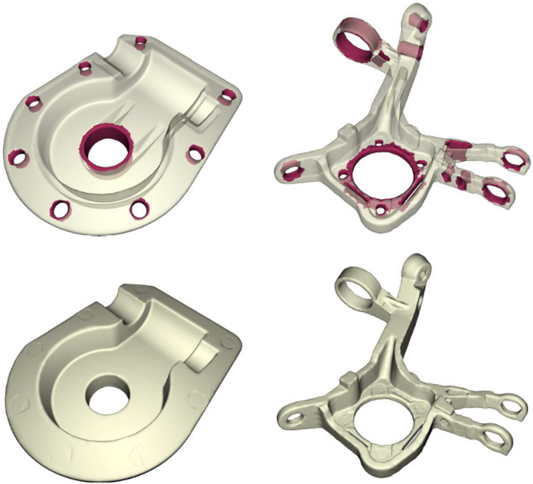
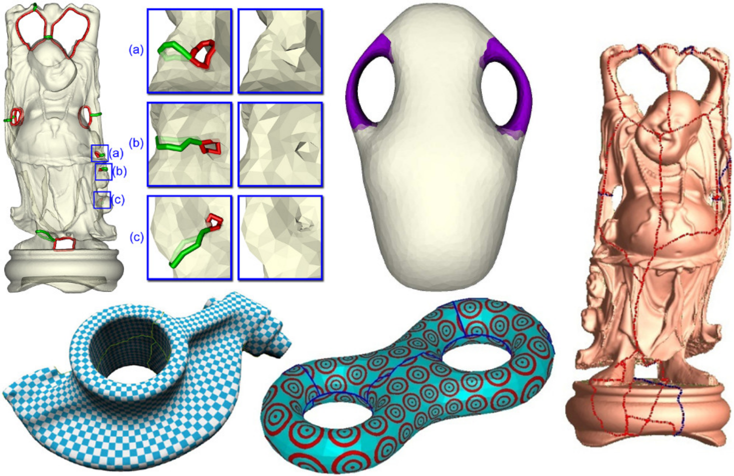
Cycles: Medical Imaging & Molecular Biology



Cycles: Computer-Aided Design



Cycles: Computer Graphics



Topological cycles: Homology

- Rank: Smith-Normal-Form; Special cases [DE95]

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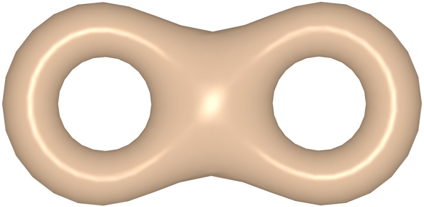

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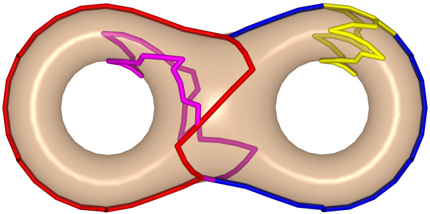

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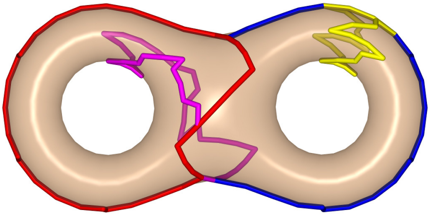

- Goal: 'Geometry-oblivious' to 'Geometry-aware'

OHBP: Optimal Homology Basis Problem

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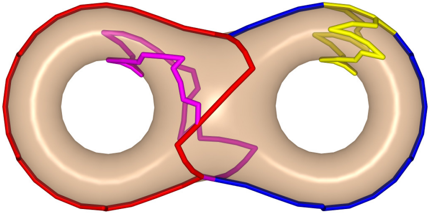

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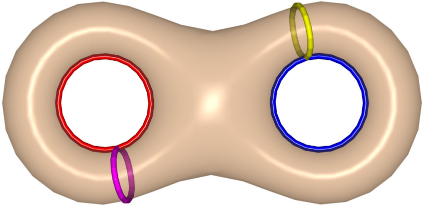

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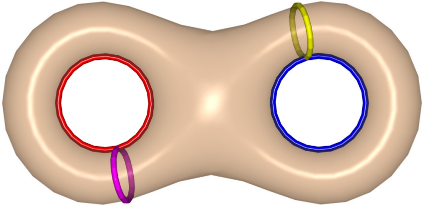

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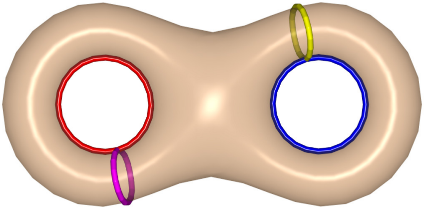

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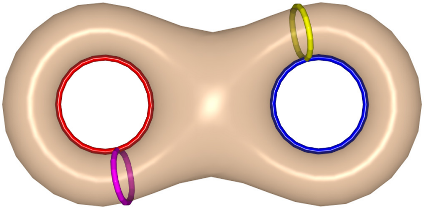

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- H_1 basis for simplicial complexes: Dey-Sun-Wang [SoCG10]

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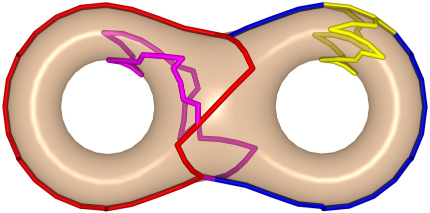

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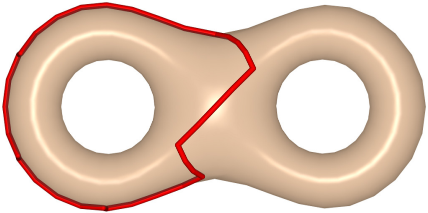

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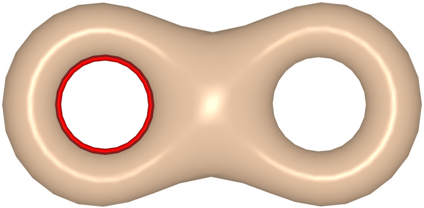

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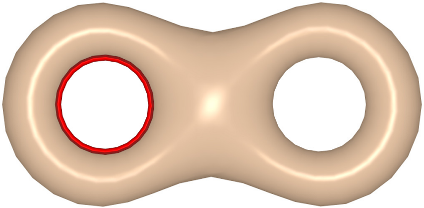

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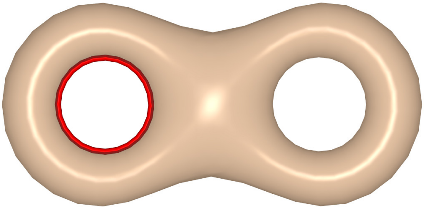

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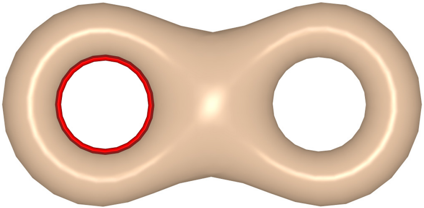

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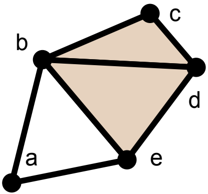

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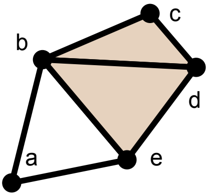

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A *p*-chain in $\mathcal{K}$ is a formal sum of p-simplices: $c = \sum_i a_i \sigma_i$; sum is the addition in a ring, $\mathbb{Z}, \mathbb{Z}_2, \mathbb{R}$ etc.

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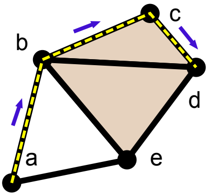

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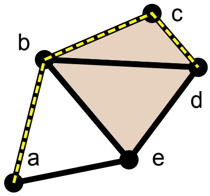

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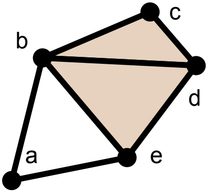

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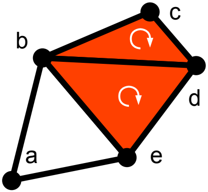

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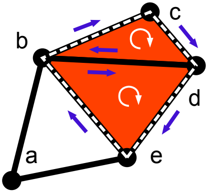

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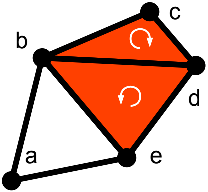

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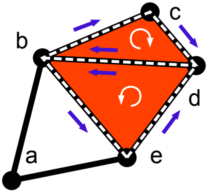

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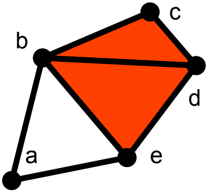

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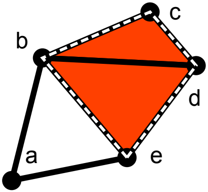

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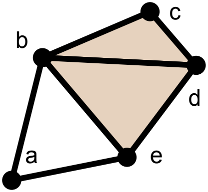

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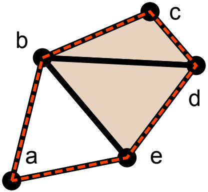

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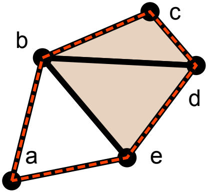

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- Each *p*-boundary is a *p*-cycle: $\partial_p \circ \partial_{p+1} = 0$

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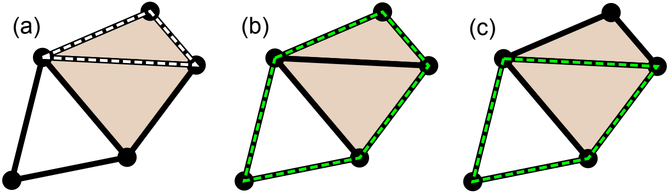
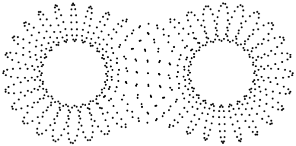
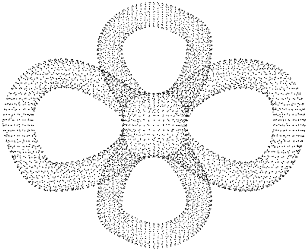


(a) trivial (null-homologous) cycle; (b), (c) nontrivial homologous cycles

PCD and simplicial complex as input





Point cloud

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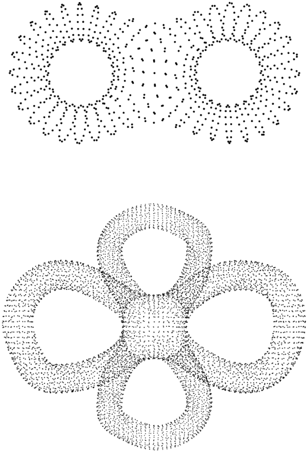

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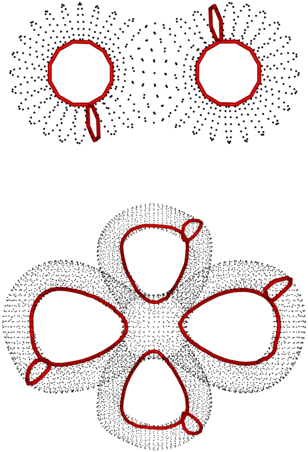
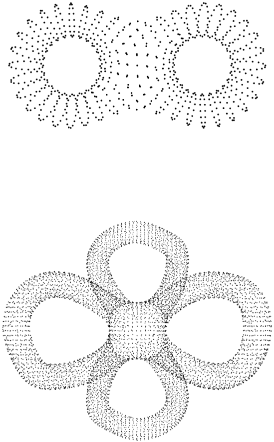


Loops

PCD→complex



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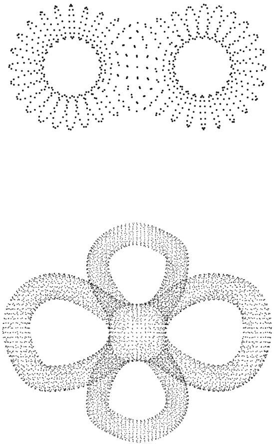

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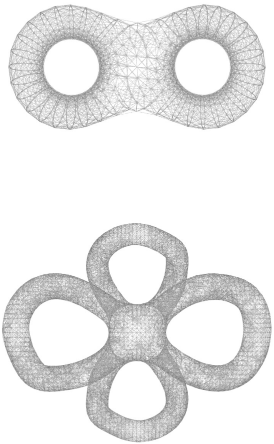

Rips complex

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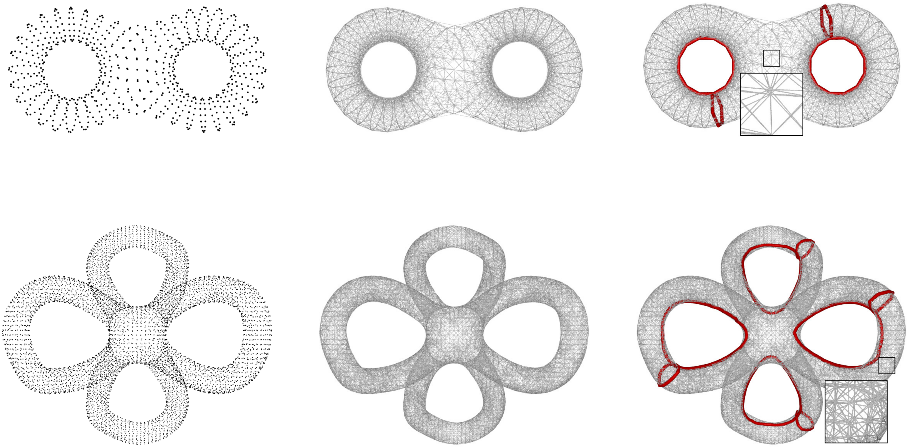

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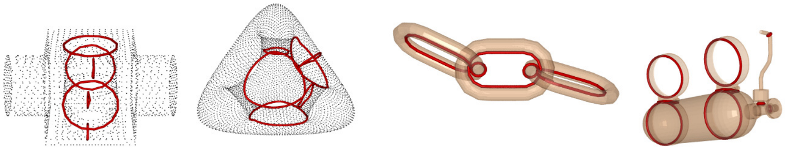

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- NP-hard for higher dimensional homology groups [CF10]

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Definition

A minimal set $\{[g_1], \ldots, [g_k]\}$ generating $\mathrm{H}_1(\mathcal{T})$ is called its basis
Here $k = \operatorname{rank} \mathrm{H}_1(\mathcal{T})$

Shortest Basis

- We associate a weight $w(g) \geq 0$ with each loop g in $\mathcal{T}$

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Definition

A shortest basis of $\mathrm{H}_1(\mathcal{T})$ is a set of k loops with minimal length that generates $\mathrm{H}_1(\mathcal{T})$

Theorem 1

Theorem

Let $\mathcal{K}$ be a finite simplicial complex with non-negative weights on edges. A shortest basis for $\mathrm{H}_1(\mathcal{K})$ can be computed in $O(n^4)$ time where $n = |\mathcal{K}|$

Approximation from Point Cloud

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- Compute a shortest basis of $\mathrm{H}_1(\mathcal{K})$
- Argue that if P is *dense*, a subset of computed loops approximate a shortest basis of $\mathrm{H}_1(\mathcal{M})$ within constant factors

Complexes

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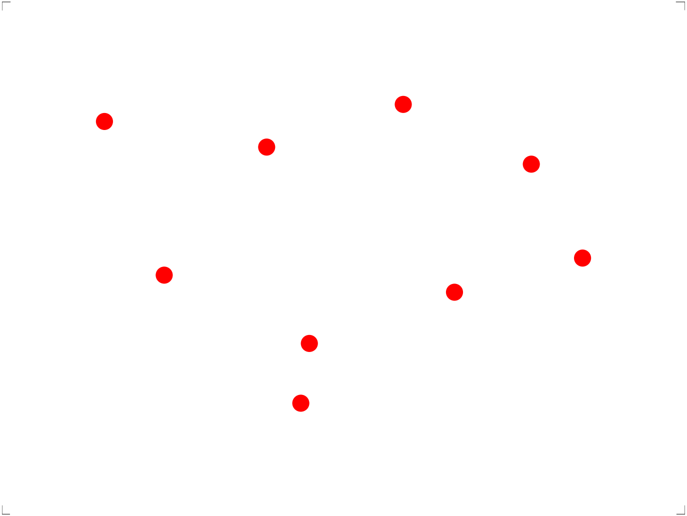
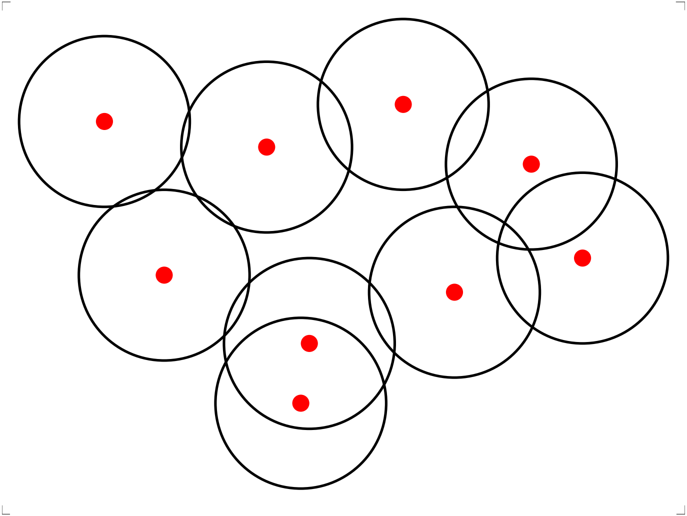
Proposition

For any finite set $P \subset \mathbb{R}^d$ and any $r \geq 0$, $\mathcal{C}^r(P) \subseteq \mathcal{R}^r(P) \subseteq \mathcal{C}^{2r}(P)$

Point set P



Balls $B(p, r/2)$ for $p \in P$



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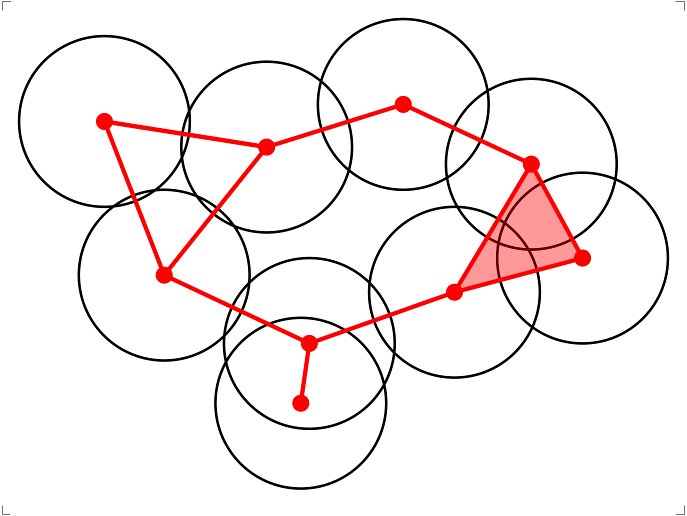

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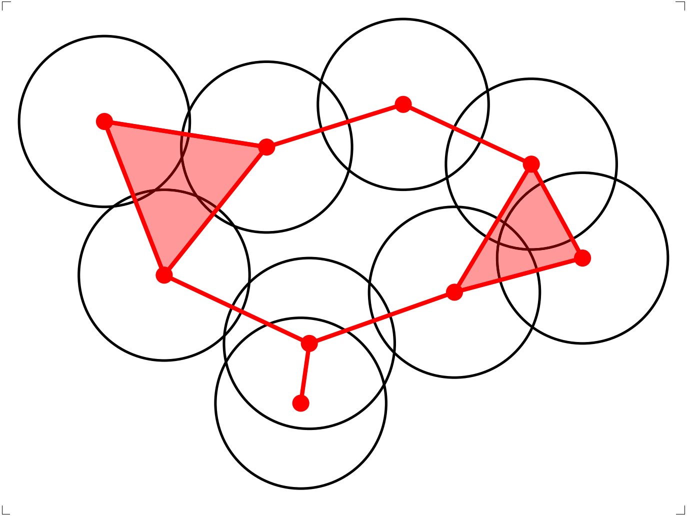

Approximation Theorem

Theorem

Let $\mathcal{M} \subset \mathbb{R}^d$ *be a smooth, closed manifold with* l *as the length of a shortest basis of* $\mathrm{H}_1(\mathcal{M})$ *and* $k = \operatorname{rank} \mathrm{H}_1(\mathcal{M})$.
Given a set $P \subset \mathcal{M}$ *of* n *points which is an* ε*-sample of* $\mathcal{M}$ *and* $4\varepsilon \leq r \leq \min\{\frac{1}{2}\sqrt{\frac{3}{5}}\rho(\mathcal{M}), \rho_c(\mathcal{M})\}$, *one can compute a set of loops* G *in* $O(nn_e^2 n_t)$ *time where*

$$\frac{1}{1 + \frac{4r^2}{3\rho^2(\mathcal{M})}} l \leq \mathrm{Len(G)} \leq (1 + \frac{4\varepsilon}{\mathrm{r}})\mathrm{l}.$$

Here n_e, n_t *are the number of edges and triangles in* $\mathcal{R}^{2r}(P)$

Optimal Homologous Cycle Problem: Our Goal

- How to compute a shortest cycle in a given class?

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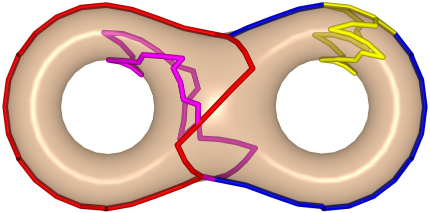

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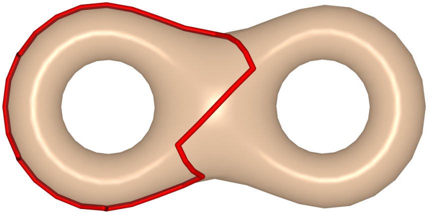

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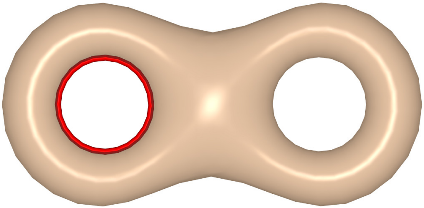

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- We characterize the complexes for which this is true
- For such complexes, the optimal cycle can be computed in polynomial time ☺

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Theorem

Let A be an $m \times n$ totally unimodular matrix and $\mathbf{b}$ an integral vector, i.e. $\mathbf{b} \in \mathbb{Z}^m$. Then the polyhedron $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0\}$ is integral meaning that $\mathcal{P}$ is the convex hull of the integral vectors contained in $\mathcal{P}$. In particular, the extreme points (vertices) of $\mathcal{P}$ are integral. Similarly the polyhedron $\mathcal{Q} = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} \geq \mathbf{b}\}$ is integral.

Optimization

- Consider an integral vector $\mathbf{b} \in \mathbb{Z}^m$ and a real vector $\mathbf{f} \in \mathbb{R}^n$.

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$$\begin{aligned} & \min \mathbf{f}^T \mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0 \\ \text{and} \quad & \mathbf{x} \in \mathbb{Z}^n. \end{aligned}$$

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Corollary

Let A be a totally unimodular matrix. Then the integer linear program above can be solved in time polynomial in the dimensions of A.

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- A p-chain $\sum_{i=0}^{m-1} x_i \sigma_i$ is defined by its coefficient vector $\mathbf{x} \in \mathbb{Z}^m$.

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The weighted 1-norm of $\mathbf{v}$ is $||W\mathbf{v}||_1$, where W is $m \times m$ diagonal matrix.

- Given a p-chain $\mathbf{c}$ and a matrix W, we need to find a chain $\mathbf{c}^*$ which has the minimal 1-norm $||W\mathbf{c}^*||$ among all chains homologous to $\mathbf{c}$

Central Idea

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- Find the conditions under which the constraint matrix of the program is totally unimodular.
- For this class of problems, relax the integer linear program to a linear program by dropping the constraint that the variables be integral.

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- Assume that $\mathcal{K}$ contains m p-simplices and n $(p+1)$-simplices.

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Program

$$\begin{aligned} &\min \|W\mathbf{x}\|_1 \\ \text{such that} \quad & \mathbf{x} = \mathbf{c} + [\partial_{p+1}]\mathbf{y} \\ \text{and} \quad & \mathbf{x} \in \mathbb{Z}^m, \mathbf{y} \in \mathbb{Z}^n. \end{aligned}$$

Integer Linear Program

Program

$$\begin{aligned} &\min \sum_i |w_i|(x_i^+ + x_i^-) \\ \text{subject to} \quad & \mathbf{x}^+ - \mathbf{x}^- = \mathbf{c} + [\partial_{p+1}]\mathbf{y} \\ & \mathbf{x}^+, \mathbf{x}^- \geq 0 \\ & \mathbf{x}^+, \mathbf{x}^- \in \mathbb{Z}^m, \mathbf{y} \in \mathbb{Z}^n. \end{aligned}$$

Linear Program

Program

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Lemma

If $B = [\partial_{p+1}]$ is totally unimodular then so is $[I \ -I \ -B \ B]$.

Theorem

If the boundary matrix $[\partial_{p+1}]$ of a finite simplicial complex of dimension greater than p is totally unimodular, the optimal homologous chain problem for p-chain can be solved in polynomial time.

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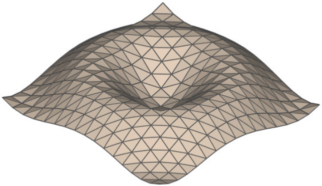

Total Unimodularity and Relative Torsion

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Theorem

$[\partial_{p+1}]$ is totally unimodular if and only if $\mathrm{H}_p(\mathcal{L}, \mathcal{L}_0)$ is torsion-free, for all pure subcomplexes $\mathcal{L}_0, \mathcal{L}$ of $\mathcal{K}$ of dimensions p and $p+1$, respectively, where $\mathcal{L}_0 \subset \mathcal{L}$. Hence, OHCP for p-chains in such complexes are polynomial time solvable by linear programs.

A Special Case

Theorem

Let $\mathbb{K}$ be a finite simplicial complex embedded in $\mathbb{R}^{d+1}$. Then, $\mathrm{H}_d(\mathcal{L}, \mathcal{L}_0)$ is torsion-free for all pure subcomplexes $\mathcal{L}_0$ and $\mathcal{L}$ of dimensions d and $d+1$ respectively, such that $\mathcal{L}_0 \subset \mathcal{L}$.

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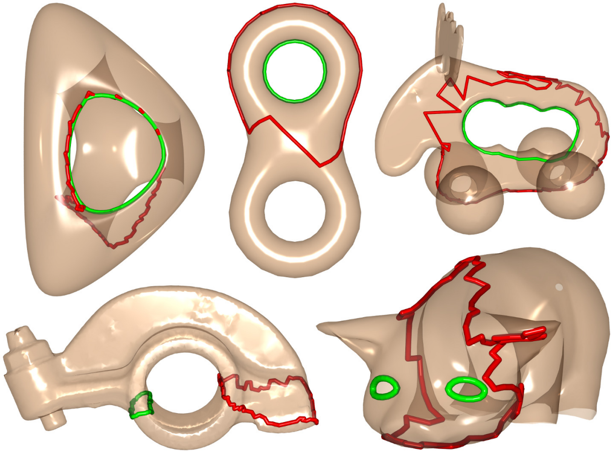
Corollary

Given a d-chain $\mathbf{c}$ *in a weighted finite simplicial complex embedded in $\mathbb{R}^{d+1}$, an optimal chain homologous to* $\mathbf{c}$ *can be computed by a linear program.*

Computed Optimal Cycles



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- What about efficient updates?

Thank You