On counting triangulations in d dimensions

Tamal Krishna Dey

Department of Computer Science, Indiana-Purdue University at Indianapolis, Indianapolis, IN 46202, USA

Abstract

Given a set of n labeled points on S^d, how many combinatorially different geometric triangulations for this point set are there? We show that the logarithm of this number is at most some positive constant times $n^{\lfloor d/2 \rfloor +1}$. Evidence is provided that for even dimensions d the bound can be improved to some constant times $n^{d/2}$.

1. Introduction

In this paper we consider the problem of counting the number of combinatorially different geometric triangulations of a fixed set of n labeled points on S^d, the d-dimensional sphere. By this we mean a triangulation consisting of geometric simplices rather than topological or combinatorial generalizations thereof. Precise definitions are given in Section 2. Let $s_d(n)$ denote the maximum number of geometric triangulations with any fixed set P of n labeled points in S^d. A more general type of triangulations, often considered in the literature, consists of topological simplices in S^d. Let $t_d(n)$ denote the maximum number of topological triangulations of any fixed set of n labeled points in S^d. Every geometric triangulation of S^d is also a topological triangulation. Therefore $s_d(n) \leqslant t_d(n)$.

Using a result of Goodman and Pollack [4], the bounds for a fixed point set can be extended to cover all point sets of some fixed cardinality. More specifically, they show that there is a positive constant $c = c(d)$ so that the logarithm of the number of combinatorially different sets of n points in S^d is at most $cn \log n$. It appears that the dominant factor in the total number of triangulations is the

number of triangulations of a single point set rather than the number of different point sets. Kalai [5] proves that for fixed d, the logarithm of the number of topological triangulations for n labeled points (not necessarily fixed) in S^d has a lower bound of $c_1 n^{\lfloor d/2 \rfloor}$ and an upper bound of $c_2 n^{\lceil d/2 \rceil} \log n$, where c_1 and c_2 are some positive constants. This implies an upper bound of $cn^{\lceil d/2 \rceil} \log n$ for $\log s_d(n)$. In general we will use c with or without index for positive constants.

Another quantity related to $s_d(n)$ is $r_d(n)$, the maximum number of geometric triangulations of n fixed and labeled points in $\mathbb{R}^d$, the d-dimensional real space. It is fairly easy to establish a correspondence between geometric triangulations in S^d and $\mathbb{R}^d$ that implies $r_d(n) \leqslant s_d(2n)$, see Section 2.

This paper is organized as follows: Section 2 introduces the basic definitions; Section 3 presents an observation about intersecting simplices that is used to prove $\log s_d(n) \leqslant cn^{\lceil d/2 \rceil}$ when d is odd. For even d we generalize a technique inspired by the work of [1] where it was used to prove that $\log s_2(n) \leqslant cn$. This technique relies on a result that is known to be true in dimension $d = 2$ and which is conjectured to hold for all constant even dimensions. Contingent upon this conjecture, we prove that $\log s_d(n) \leqslant cn^{d/2}$ for even d.

2. Definitions

Think of S^d as the unit sphere in $\mathbb{R}^{d+1}$ centered at the origin, o. A *hemisphere* of S^d is the intersection of S^d with a closed halfspace in $\mathbb{R}^{d+1}$ whose bounding hyperplane contains o. Any collection V of k points in S^d is *a.i.* if $V \cup \{o\}$ is affinely independent in $\mathbb{R}^{d+1}$. V defines a unique *great sphere* in S^d, namely the intersection of S^d with the affine hull of $V \cup \{o\}$. If V is a.i. then this great sphere is a $(k-1)$-sphere of S^d. For $0 \leqslant k \leqslant d$, a *spherical polytope* in S^d is the intersection of finitely many hemispheres. It is a k-polytope if it contains $k+1$ a.i. points (vertices of the polytope) but not $k+2$. In what follows, we assume the points in P are in general position. By this we mean that no hemisphere contains P and any $d+1$ points of P are a.i.

A *spherical k-simplex* in S^d is the intersection of all hemispheres that contain some set of $k+1 \leqslant d+1$ points, the vertices of the simplex. Thus, any set V of $k+1 \leqslant d+1$ a.i. points in S^d defines a unique spherical k-simplex, $\Delta = \Delta_V$. For $0 \leqslant j \leqslant k$, a *$j$-face* of Δ is the spherical j-simplex defined by any $j+1$ of the $k+1$ vertices of Δ. Let $\Delta_1 = \Delta_{V_1}$ be a spherical k-simplex and $\Delta_2 = \Delta_{V_2}$ be a spherical l-simplex. We say that Δ_1 and Δ_2 *intersect improperly* if $\operatorname{ri}(\Delta_1) \cap \operatorname{ri}(\Delta_2) \neq \emptyset$ where $\operatorname{ri}(X)$ denotes the relative interior of X. If the $k+l+2$ vertices in $V_1 \cup V_2$ are a.i. then Δ_1 and Δ_2 intersect improperly if and only if $\Delta_1 \cap \Delta_2$ is not a face of both. Furthermore, we say that Δ_1 and Δ_2 *cross* if they intersect improperly and $V_1 \cap V_2 = \emptyset$. For P a finite set of point in general position in S^d, we denote by $\binom{P}{k}$ the set of all spherical $(k-1)$-simplices with vertices in P. A subset $T \subseteq \binom{P}{k}$ is *crossing-free* if no two spherical $(k-1)$-simplices in T cross. A *geometric*

triangulation of P is defined by a collection of spherical d-simplices Δ_{V_i} so that:

(i) $\Delta_{V_i} \cap P = V_i$, for each i,

(ii) no two d-simplices intersect improperly, and

(iii) the union of the d-simplices is S^d.

Conditions (i) and (ii) require that the collection of spherical d-simplices form a simplicial cell complex, and (iii) requires that S^d is the underlying space of the complex.

Similar definitions are possible in $\mathbb{R}^d$. A set of $k + 1 \leq d + 1$ affinely independent points defines a unique k-simplex, namely the convex hull of the $k + 1$ points. Alternatively, this k-simplex can be defined as the intersection of all closed half-spaces that contain the $k + 1$ points. A *geometric triangulation* of a finite point set $P \subseteq \mathbb{R}^d$ is defined by a collection of d-simplices so that each d-simplex intersects P in its vertices, no two d-simplices intersect improperly, and the union of the d-simplices is the convex hull of P. By central projection, such a triangulation in $\mathbb{R}^d$ can be mapped to the southern hemisphere of S^d where it forms a partial triangulation of S^d. Let P' be the set of vertices of this partial triangulation. For reasons stated below, we give another transformation to this projected triangulation. Keeping the southern pole fixed, we grow S^d until all circumscribing spheres of the d-simplices with vertices in P' remain solely within the southern hemisphere. To make it a unit sphere, we shrink the enlarged sphere centrally. Let P' be transformed to P'_t by this process. The transformed triangulation with the vertex set P'_t still constitutes a partial triangulation of S^d. To complete this triangulation we also project the triangulation from $\mathbb{R}^d$ on the northern hemisphere and transform it analogously. Let P''_t be the corresponding vertex set. The two partial triangulations can be connected by considering the convex hull of $P'_t \cup P''_t$ in $\mathbb{R}^{d+1}$. Any face of the convex hull that has vertices in P'_t as well as in P''_t can now be mapped to a spherical simplex that connects the two partial triangulations. Due to the above transformations, it is guaranteed that these faces connect the two partial triangulations properly without piercing any of their simplices. Given the triangulation in $\mathbb{R}^d$, this construction implies a unique triangulation of S^d. Therefore $r_d(n) \leq s_d(2n)$.

A topological triangulation of S^d is the geometric realization of an abstract simplicial complex on S^d. The simplices in such triangulation are curved arbitrarily and are not necessarily *spherical* according to our definitions. Obviously, every geometric triangulation of S^d is also a topological triangulation. Thus, $r_d(n) \leq s_d(2n) \leq t_d(2n)$. Kalai [5] proved that $c_1 n^{\lfloor d/2 \rfloor} \leq \log t_d(n) \leq c_2 n^{\lfloor d/2 \rfloor} \log n$. Any asymptotic upper bound on $t_d(n)$ also applies to $r_d(n)$ and $s_d(n)$. However, the same is not true for lower bounds. In this paper, we improve the asymptotic upper bound on $s_d(n)$ and hence also on $r_d(n)$. It remains open to prove nontrivial lower bounds on $r_d(n)$ and $t_d(n)$. We suspect that the current asymptotic lower bound on $t_d(n)$ also applies to $r_d(n)$ and $s_d(n)$, and it is tight. This is known to be true for $d = 2$ [1].

3. Simplex crossings in S^d

Given two spherical simplices that intersect improperly, we prove that there is a lower dimensional face of one that crosses a higher dimensional face of the other. In what follows, by a simplex we mean a spherical simplex and by a triangulation we means a geometric triangulation in S^d.

Lemma 3.1. *For $k_1 + k_2 \geqslant d$, let Δ_1 be a k_1-simplex that intersects improperly a k_2-simplex Δ_2 in S^d. There must be a $\lfloor d/2 \rfloor$-face of one simplex that crosses the other simplex.*

Proof. Actually we prove a stronger statement. Let $k_1 + k_2 \geqslant d$. Then there is an l-face of Δ_1 that crosses an l_2-face of Δ_2, with $l_1 + l_2 = d$. This clearly implies that one of l_1 and l_2 is less than or equal to $\lfloor d/2 \rfloor$.

Let V_i be the vertex set of Δ_i, for $i = 1, 2$, and define $m = k_1 + k_2 - d$. First note that $|V_1 \cap V_2| \leqslant m$. Otherwise, $|V_1 \cup V_2| = (k_1 + k_2 + 2) - |V_1 \cap V_2| \leqslant d + 1$, and by general position assumption Δ_1 and Δ_2 cannot have an improper intersection.

Again by general position assumption the great spheres defined by V_1 and V_2 intersect in an m-sphere. By definition of improper intersection, $Q = \Delta_1 \cap \Delta_2$ is therefore a spherical m-polytope. It has at least one vertex $u \notin V_1 \cap V_2$, for otherwise Q would be contained in the simplex defined by the shared vertices. This simplex is disjoint from the relative interiors of Δ_1 and Δ_2, or else $m = d$ which can be the case only if $V_1 = V_2$. But this possibility is excluded in the definition of improper intersection. Let l_1 and l_2 be minimal so that u belongs to the intersection of an l_1-face Δ_1' of Δ_1 and an l_2-face Δ_2' of Δ_2. Since the dimension of u is 0, we have $(k_1 - l_1) + (k_2 - l_2) = m$, and thus $l_1 + l_2 = k_1 + k_2 - m = d$. Furthermore, Δ_1' and Δ_2' are vertex disjoint because they have altogether only $d + 2$ vertices and if some are shared then $\Delta_1' \cap \Delta_2'$ would be the simplex defined by the shared vertices. This contradicts $u \notin V_1 \cap V_2$. $\square$

From the above Lemma we have the following simple observation about triangulations in S^d. We observe that all higher dimensional faces of a triangulation can be completely determined from its $\lfloor d/2 \rfloor$-faces as follows. To enumerate all k-faces of the triangulation, $k > \lfloor d/2 \rfloor$, from all possible k-faces out of given $\lfloor d/2 \rfloor$-faces. Retain only those k-faces that do not intersect any given $\lfloor d/2 \rfloor$-face. These are the k-faces of the triangulation. This is true because any k-face of the triangulation must have $\lfloor d/2 \rfloor$-faces from the given set of $\lfloor d/2 \rfloor$-faces and any k-face that is not in the triangulation must intersect another k-face of the triangulation and hence a $\lfloor d/2 \rfloor$-face of the triangulation due to Lemma 3.1. This observation should be compared with the result of Dancis [2], who shows that triangulated d-manifolds are completely determined by their $\lfloor d/2 \rfloor + 1$-faces.

Lemma 3.2. $\log s_d(n) = O(n^{\lfloor d/2 \rfloor + 1})$.

Proof. By above observation, any triangulation of n fixed labeled points in S^d can be completely determined by the set of $\lfloor d/2 \rfloor$-faces of the triangulation. There can be at most $2^{O(n^{\lfloor d/2 \rfloor + 1})}$ different such sets. $\square$

Note that combining Lemma 3.2 with the result of Kalai [5], we get $\log s_d(n) = O(n^{\lceil d/2 \rceil})$ for odd dimensions and $\log s_d(n) = O(n^{\lceil d/2 \rceil} \log n)$ for even dimensions. The $\log n$ factor in the bound for even dimensions seems unnatural. We show that $\log s_d(n) = O(n^{d/2})$ for even d, if we assume the following conjecture. In what follows we assume d is even and $u = d/2$.

Conjecture 3.1. Let T be a set of crossing free u-simplices with n vertices in S^d. Then $|T| = O(n^u)$.

Clearly, $|T| = O(n^{u+1})$, and it is known that $|T| = O(n^u)$ if T forms a subcomplex of a topological triangulation of S^d [6]. Furthermore, a recent result of Živaljević [7] implies that $|T| = O(u^{u+1-\varepsilon})$ where $\varepsilon = (\frac{1}{3})^u$. These results suggest that it is unlikely that the above conjecture is false.

Note that, for even d two u-simplices in S^d can intersect only in a point. This implies that improper intersection and crossing mean the same thing when d is even. The following Lemma establishes an important fact about the number of u-simplex crossings in a set of t u-simplices with n vertices. Let P be a set of n points in S^d and $x^{(d)}(P, T)$ denote the maximum number of u-simplex crossings in a set T of t u-simplices with vertices in P. Define $x^{(d)}(n, t) = \min_{|P|=n, |T|=t}(P, T)$. The next lemma which implies

$$x^{(d)}(n, t) = \Omega\left(\frac{t^\gamma}{n^{(\gamma-2)(u+1)}}\right)$$

is a generalization of a similar result in three dimensions [3].

Lemma 3.3. *If the maximum size of any set of crossing free u-simplices with n vertices is $c_1 n^{u+1-\delta}$ (for some constant $0 < \delta \leq 1$) then there exists a constant c_2 so that*

$$x^{(d)}(n, t) \geq c_2 \binom{n}{2u+2}\left(\frac{t}{\binom{n}{u+1}}\right)^\gamma$$

where $t \geq c_3 n^{u+1-\delta}$, $\gamma = 1 + (u+1)/\delta$, and $c_3 = c_1 + 1$.

Proof. Let T be a set of t u-simplices with n vertices in S^d that realizes $x^{(d)}(n, t)$.

We show by induction that there is a small enough constant c_2 so that

$$x^{(d)}(n, t) \geq c_2 \binom{n}{2u+2} \left(\frac{t}{\binom{n}{u+1}} \right)^\gamma. \tag{1}$$

Let $\alpha = u + 1 - \delta$. From our assumption, there can be at most $c_1 n^\alpha$ u-simplices with n vertices in S^d that are crossing free. We assume that $t \geq c_3 n^\alpha$ where $c_3 = c_1 + 1$. Certainly, there are at least $t - c_1 n^\alpha$ pairwise intersections occurring between u-simplices that are vertex disjoint. This is because for each extra element over $c_1 n^\alpha$, we have at least one crossing. The fact of vertex disjointness is important for the inductive step.

Base cases

  Case 1: $n \leq n_0$ for some fixed $n_0 > 2u + 2$.
  By assumption, $x^{(d)}(n, t) \geq t - c_1 n^\alpha \geq 1$. We can make

$$c_2 \binom{n}{2u+2} \left(\frac{t}{\binom{n}{u+1}} \right)^\gamma \leq 1 \tag{2}$$

by choosing c_2 sufficiently small since for $n \leq n_0$, $t \leq c$ is a constant.
  Case 2: $c_3 n^\alpha < t \leq (c_3 + 1) n^\alpha$, and $n > n_0$.
  We have to prove that

$$c_2 \binom{n}{2u+2} \left(\frac{t}{\binom{n}{u+1}} \right)^\gamma \leq t - c_1 n^\alpha. \tag{3}$$

Since $n > 2u + 2$, $\binom{n}{u+1} \geq c_4 n^{u+1}$ for some constant c_4.
Thus L.H.S. of 3 is less than or equal to

$$c_2 \frac{n^{2u+2}(c_3 + 1)^\gamma n^{\alpha\gamma}}{c_4^\gamma n^{(u+1)(\gamma)}} = c_5 n^\alpha$$

where $c_5 = c_2(c_3 + 1)^\gamma / c_4^\gamma$, a constant. R.H.S. of 3 is greater than or equal to $c_3 n^\alpha - c_1 n^\alpha = n^\alpha$. Thus we have to show that $c_5 n^\alpha \leq n^\alpha$, which is true if c_2 is chosen sufficiently small.
  Inductive step: $t > (c_3 + 1) n^\alpha$, and $n > n_0$.
  Let $T(w)$ represent the set of u-simplices in T that are not incident on the vertex w and let $t(w) = |T(w)|$. For each crossing between two vertex disjoint u-simplices Δ_1 and Δ_2, we count all vertices except the ones of Δ_1 and Δ_2. Alternatively, we can think of this count as the sum of all nontrivial intersections between u-simplices in $T(w)$ for each vertex w. Thus we have

$$(n - 2u - 2)x^{(d)}(n, t) = \sum_{w \in V} x^{(d)}(n - 1, t(w))$$

$$\geq c_2 \binom{n-1}{2u+2} \left(\frac{\sum_{w \in V} t(w)^\gamma}{\binom{n-1}{u+1}^\gamma} \right) \quad \text{by induction.}$$

Now $\sum_{w \in V} t(w) = (n - u - 1)t$. Thus

$$\sum_{w \in V} t(w)^\gamma \geq n \left(\frac{(n - u - 1)t}{n} \right)^\gamma.$$

This gives

$$x^{(d)}(n, t) \geq c_2 \frac{n}{n - 2u - 2} \binom{n - 1}{2u + 2} \left(\frac{(n - u - 1)t}{n \binom{n - 1}{u + 1}} \right)^\gamma$$

$$= c_2 \binom{n}{2u + 2} \left(\frac{t}{\binom{n}{u + 1}} \right)^\gamma. \qquad \square$$

Applying the pigeon-hole principle on the lower bound of $x^{(d)}(n, t)$ we infer that there is at least one u-simplex that intersects many other u-simplices. This is stated in the following lemma.

Lemma 3.4. *Let T be a set of u-simplices in S^d. There exists a u-simplex that intersects*

$$\Omega \left(\frac{t^{\gamma - 1}}{n^{(\gamma - 2)(u + 1)}} \right)$$

other u-simplices where $|T| = t > c_3 n^{u + 1 - \delta}$, and n is the size of the vertex set.

4. Crossing free simplices

Conjecture 3.1 implies that $\delta = 1$. Using this in Lemma 3.4 we establish that there exists a u-simplex in T that intersects

$$\Omega \left(\frac{t^{u + 1}}{n^{u(u + 1)}} \right)$$

u-simplices. Using this fact, we deduce that for even d there are at most $2^{O(n^u)}$ crossing free sets of u-simplices with n fixed vertices in S^d. Define $F(t)$ as the largest number of crossing free subsets of u-simplices that can be chosen from t u-simplices in S^d with n fixed vertices. Since the set of u-simplices of a triangulation completely determines it, an upper bound on $F(t)$ for $t = \binom{n}{u + 1}$ also gives an upper bound on the number of triangulations with n vertices in S^d.

Lemma 4.1. *Assuming Conjecture 3.1, $F(t) = 2^{O(n^u)}$ for any even d.*

Proof. Let c be large enough so that there is a u-simplex that crosses at least

$$\frac{(u + 1)t^{u + 1}}{cn^{u(u + 1)}}$$

other u-simplices if $t > cn^u \geqslant c_3 n^u$. Assuming conjecture 3.1, we can always find such a u-simplex due to Lemma 3.4.

Case 1: $t \leqslant cn^u$.

In this case we have $F(t) \leqslant 2^t \leqslant 2^{cn^u}$.

Case 2: $t > cn^u$.

In this case we prove that $F(t) \leqslant C^{n^u} f(t)$ where

$$C = (2c)^{\left(c + \frac{1}{c^{u-1}}\right)} \quad \text{and} \quad f(t) = \left(\frac{t}{n^u}\right)^{-\frac{cn^{u(u+1)}}{t^u}}.$$

We show later that $f(t) \leqslant 1$ for $n^u \leqslant t \leqslant \binom{n}{u+1}$ implying $F(t) = 2^{O(n^u)}$. We use induction.

Base Case: $cn^u \leqslant t \leqslant 2cn^u$.

In this case we have

$$F(t) \leqslant 2^{2cn^u} \leqslant (2c)^{cn^u}\left(\frac{t}{n^u}\right)^{\frac{cn^{u(u+1)}}{t^u}} f(t) \qquad \text{provided } c > 2$$

$$\leqslant (2c)^{cn^u}(2c)^{\frac{cn^{u(u+1)}}{c^u n^{u^2}}} f(t)$$

$$= (2c)^{\left(c + \frac{1}{c^{u-1}}\right)n^u} f(t)$$

$$\leqslant C^{n^u} f(t), \quad \text{where } C = (2c)^{\left(c + \frac{1}{c^{u-1}}\right)}.$$

Inductive step: $t \geqslant 2cn^u$.

Since there is a u-simplex that crosses at least

$$\frac{(u+1)t^{u+1}}{cn^{u(u+1)}}$$

other u-simplices, we have

$$F(t) \leqslant F(t-1) + F\left(t - \frac{(u+1)t^{u+1}}{cn^{u(u+1)}}\right).$$

Let $t = kn^u$ where $2c \leqslant k < n$.

$$t - \frac{(u+1)t^{u+1}}{cn^{u(u+1)}} = kn^u - \frac{(u+1)k^{u+1}n^{u(u+1)}}{cn^{u(u+1)}}$$

$$= kn^u - \frac{(u+1)k^{u+1}}{c}$$

$$= kn^u\left(1 - \frac{(u+1)k^u}{cn^u}\right)$$

$$> kn^u\left(1 - \frac{u+1}{c}\right)$$

$$> cn^u \quad \text{if } c > 2(u+1).$$

So we can apply the inductive assumption and get

$$F(t) \leq F(t-1) + F\left(t - \frac{(u+1)t^{u+1}}{cn^{u(u+1)}}\right)$$

$$< C^{n^u}f(t-1) + C^{n^u}f\left(t - \frac{(u+1)t^{u+1}}{cn^{u(u+1)}}\right)$$

$$< C^{n^u}f(t) \quad \text{by the property (5) of } f(t) \text{ where } t > 9^{u+1}n^u.$$

Taking c to be sufficiently large, this proves that $F(t) = 2^{O(n^u)}$ for all $t \geq 0$. $\square$

Now we show that the function f indeed has the properties used in the previous Lemma. Let

$$f(x) = \left(\frac{x}{n^u}\right)^{-\frac{cn^{u(u+1)}}{x^u}}$$

for $n^u \leq x \leq \binom{n}{u+1}$ and $c > 0$ is a sufficiently large constant.

(1) $f(x) \leq 1$ for $x \geq n^u$.

(2) $f'(x) = f(x) \dfrac{cn^{u(u+1)}}{x^{u+1}} \left\{ \ln\left(\frac{x}{n^u}\right) - u \right\}.$

Hence

$$f'(x) > \frac{cn^{u(u+1)}}{x^{u+1}} f(x) \quad \text{if } x > e^{u+1}n^u.$$

(3) $f(x) - f(x-1) = f'(y)$ for some $x - 1 \leq y \leq x$

because of the mean value theorem. Therefore

$$f(x) - f(x-1) > \frac{cn^{u(u+1)}}{x^{u+1}} f(x-1)$$

provided $x - 1 > e^{u+1}n^u$ and hence

$$f(x-1) < \frac{x^{u+1}}{x^{u+1} + cn^{u(u+1)}} f(x).$$

(4) $f\left(x - \dfrac{(u+1)x^{u+1}}{cn^{u(u+1)}}\right) \leq c' \dfrac{n^{u(u+1)}}{x^{u+1}} f(x)$

where $c' = (e^{u+1})^{2^u}$ is a constant assuming $c > 2(u+1)$. We sketch the proof below.

For $0 < m < 1$, $f(x(1-m)) = a(x, m)b(x, m)$ where

$$a(x, m) = \left(\frac{x}{n^u}\right)^{-\frac{cn^{u(u+1)}}{x^u(1-m)^u}}$$

and

$$b(x, m) = (1 - m)^{-\frac{cn^{u(u+1)}}{x^u(1-m)^u}}.$$

Since $1/(1 - m) > 1 + m$, we have

$$a(x, m) < \left(\frac{x}{n^u}\right)^{-\frac{c(1+m)^u n^{u(u+1)}}{x^u}}$$

$$< \left(\frac{x}{n^u}\right)^{-\frac{c(1+m)n^{u(u+1)}}{x^u}}$$

$$< f(x)\left(\frac{x}{n^u}\right)^{-\frac{cmn^{u(u+1)}}{x^u}} \qquad \text{for } x > n^u$$

Also $1 + y \le e^y$. We have

$$b(x, m) < (e^m)^{\frac{cn^{u(u+1)}}{x^u(1-m)^u}}.$$

$$f\left(x - \frac{(u+1)x^{u+1}}{cn^{u(u+1)}}\right) = a\left(x, \frac{(u+1)x^u}{cn^{u(u+1)}}\right)b\left(x, \frac{(u+1)x^u}{cn^{u(u+1)}}\right)$$

$$a\left(x, \frac{(u+1)x^u}{cn^{u(u+1)}}\right) \le \frac{n^{u(u+1)}}{x^{u+1}}f(x)$$

$$b\left(x, \frac{(u+1)x^u}{cn^{u(u+1)}}\right) \le e^{\frac{u+1}{(1-m)^u}}\left(m = \frac{(u+1)x^u}{cn^{u(u+1)}} < \frac{u+1}{c}\right)$$

$$< (e^{u+1})^{\left(\frac{c}{c-u-1}\right)^u}$$

$$< (e^{u+1})^{2^u} = c', \quad \text{a constant if } c > 2(u+1)$$

$$(5) \qquad f(x - 1) + f\left(x - \frac{(u+1)x^{u+1}}{cn^{u(u+1)}}\right) < f(x)$$

for $x > kn^u$ where k is some constant determined as follows. By (3) and (4) we have to show that

$$\frac{x^{u+1}}{x^{u+1} + cn^{u(u+1)}} + \frac{c'n^{u(u+1)}}{x^{u+1}} \le 1$$

Let $x = kn^u$, where $2c \le k < n$. We show that the above relation is satisfied for $k > 9^{u+1}$. We must have

$$c'cn^{2u(u+1)} + c'x^{u+1}n^{u(u+1)} \le cx^{u+1}n^{u(u+1)}$$

$$c'c + c'k^{u+1} \le ck^{u+1}$$

$$c'c \le k^{u+1}(c - c')$$

$$k > \left(\frac{c'c}{c - c'}\right)^{\frac{1}{u+1}}$$

$$< (2c')^{\frac{1}{u+1}} \quad \text{if } c > 2c'$$

$$< 2^{\frac{1}{u+1}}e^{2^u}$$

$$< 9^{u+1}.$$

Combining the results of Lemma 3.2 and Lemma 4.1 we get the following result.

Theorem 4.2. $\log s_d(n) = O(n^{\lceil d/2 \rceil})$ *when d is odd. Further, assuming Conjecture 3.1,* $\log s_d(n) = O(n^{d/2})$ *when d is even.*

Acknowledgement

References

[1] M. Ajtai, V. Chvátal, M.M. Newborn and E. Szemerédi, Crossing-free subgraphs, Ann. Discrete Math. 12 (1982) 9–12.

[2] J. Dancis, Triangulated n-manifolds are determined by their $\lfloor n/2 \rfloor + 1$-skeletons, Topology Appl. 18 (1984) 17–26.

[3] T.K. Dey and H. Edelsbrunner, Counting triangle crossings and halving planes, Ninth Ann. Sympos. Comput. Geom. (1993) 270–273.

[4] J.E. Goodman and R. Pollack, Upper bounds for configurations and polytopes in $\mathbb{R}^d$, Discrete Comput. Geom. (1986) 219–227.

[5] G. Kalai, Many triangulated spheres, Discrete Comput. Geom. 3 (1988) 1–14.

[6] R. Stanley, The upper-bound conjecture and Cohen-Macaulay rings, Stud. Appl. Math. 54 (1975) 135–142.

[7] R. Živaljević, New cases of the colored Tverberg's theorem, Personal communications.