Persistence for Circle Valued Maps

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Abstract

We study circle valued maps and consider the *persistence of the homology of their fibers*. The outcome is a finite collection of computable invariants (bar codes and Jordan cells) which answer the basic questions on persistence and in addition encode the topology of the source space and its relevant subspaces. We show how to recover the homology of the source space and of its relevant subspaces and how to compute the invariants. In particular, we reduce the computation of the bar codes to algorithms described for zigzag [4] and standard persistence [11, 16]. We show how persistence of circle valued maps can be extended to determine persistence for a class of 1-cocycles.

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1 Introduction

Data analysis provides plenty of scenarios where one ends up with a nice space, most often a simplicial complex, a smooth manifold, or a stratified space equipped with a real valued or a circle valued map. The persistence theory, introduced in [11] and refined in [16], provides a great tool for analyzing real valued maps in terms of topological invariants. Similar theory for circle valued maps has not yet been developed in the literature. The work in [17] brings the concept of circle valued maps in the context of persistence by deriving a circle valued map for a given data using the existing persistence theory. In contrast, we develop a persistence theory for circle valued maps.

One place where circle valued maps appear naturally is the area of dynamics of vector fields. The measurements in the dynamics described by a curl free vector field [1] can be interpreted as 1-cocycles which are intimately connected to circle valued maps as we show later in this paper. Consequently, a notion of persistence for circle valued maps also provides a notion of persistence for 1-cocycles. In summary, persistence theory for circle valued maps promises to play the role for vector fields as does the standard persistence theory for the scalar fields.

One of the main concepts of the persistence theory is the notion of *bar codes* [16]–invariants that characterize a scalar map at the homology level. We show that circle valued maps, when characterized at homology level, require a new invariant called *Jordan cells* in addition to the bar codes. We describe these invariants using quiver representation theory which has been used in [4] to develop zigzag persistence. The standard persistence [11, 16] which we refer as *sublevel persistence* deals with sublevel sets. The notion when extended to *level sets* provides what we refer to as *level persistence*. The zigzag persistence invariants provide complete invariants (bar codes) for level persistence using representation theory for some simple quivers (linear quivers). It turns out that representation theory of more complicated quivers (cyclic quivers) provides the complete invariants for persistence of the circle valued maps.

Our results include a derivation of the homology for the source space and its relevant subspaces in terms of the invariants (Theorem 3.1 and 3.2). The result also applies to real valued maps as they are special cases of the circle valued maps. This leads to a result (Corollary 3.3) which to our knowledge has not yet appeared in the literature.

After developing the results on invariants, we propose an algorithm to compute the bar codes. This involves, in theory, lifting the circle valued map to a cyclic covering, but, in practice, computing only a truncated version of the cyclic covering, and then applying persistence on a relevant subset of this truncated covering. For a simplicial complex, the entire computation can be done by manipulating the original matrix that encodes the input complex and the map even though the computation involves level sets and interval sets. Once the relevant matrix for a truncated covering is computed, one can use the algorithm of zigzag persistence [4] to compute the bar codes. In appendix D, we indicate how one can reduce the problem to computing the bar codes for sublevel persistence and hence take advantage of the algorithms developed in [6, 11, 16]. We lay down two approaches to calculate the Jordan cells, one in Appendix F which computes Jordan cells only and one in Appendix E which computes both the bar codes and the Jordan cells. We leave it open how to convert these approaches into efficient algorithms.

2 Definitions and background

We begin with the technical definition of tameness of a map which is essential for finite computations and elimination of pathological cases as recognized in earlier works as well [5, 12].

For a continuous map $f : X \to Y$ between two topological spaces X and Y, let $X_U = f^{-1}(U)$ for $U \subseteq Y$. When $U = y$ is a single point, the set X_y is called a *fiber* over y and is also commonly known as

[1] It is not necessary to be curl free; it suffices to have a Lyapunov closed one form

a level set. We call the continuous map $f : X \to Y$ *good* if every $y \in Y$ has a contractible neighborhood U so that the inclusion $X_y \to X_U$ is a homotopy equivalence. The continuous map $f : X \to Y$ is a *fibration* if each $y \in Y$ has a neighborhood U so that the maps $f : X_U \to U$ and $pr : X_y \times U \to U$ are fiber wise homotopy equivalent. This implies that there exists continuous maps $l : X_U \to X_y \times U$ with $pr|_U \cdot l|_U = f|_U$ which, when restricted to the fiber for any $z \in U$, are homotopy equivalences. In particular, f is good.

Definition 2.1 *A proper continuous map $f : X \to Y$ is* tame *if it is good, and for some discrete closed subset $S \subset Y$, the restriction $f : X \setminus f^{-1}(S) \to Y \setminus S$ is a fibration. The points in $S \subset Y$ which prevent f to be a fibration are called* critical values*.*

If $Y = \mathbb{R}$ or $\mathbb{S}^1$ and X is compact, then the set of critical values is finite, say $s_1 < s_2 < \cdots s_k$. The fibers above them, X_{s_i}, are referred to as *singular fibers*. All other fibers are called *regular*. In the case of $\mathbb{S}^1$, s_i can be taken as angles and we can assume that $0 < s_i \leq 2\pi$. Clearly, for the open interval (s_{i-1}, s_i) the map $f : f^{-1}(s_{i-1}, s_i) \to (s_{i-1}, s_i)$ is a fibration which implies that all fibers over angles in (s_{i-1}, s_i) are homotopy equivalent with a fixed regular fiber, say X_{t_i}, with $t_i \in (s_{i-1}, s_i)$. In particular, there exist maps $a_i : X_{t_i} \to X_{s_i}$ and $b_i : X_{t_i} \to X_{s_{i-1}}$, unique up to homotopy, derived by restricting any inverse of the homotopy equivalence $X_u \subset X_U$ to the fibers of $f : X_U \to U$. These maps determine homotopically $f : X \to Y$, when $Y = \mathbb{R}$ or $\mathbb{S}^1$. For simplicity in writing, when $Y = \mathbb{R}$ we put $t_{k+1} \in (s_k, \infty)$ and $t_1 \in (-\infty, s_1)$ and when $Y = \mathbb{S}^1$ we put $t_{k+1} = t_1 \in (s_k, s_1 + 2\pi)$.

Note that all scalar or circle valued simplicial maps on a simplicial complex, and smooth maps with generic isolated critical points on a smooth manifold or stratified space are tame. In particular, Morse maps are tame. For the tame maps in this paper we will require an additional property that the space X is compact.

2.1 Persistence and invariants

Since our goal is to extend the notion of persistence from real valued maps to circle valued maps, we first summarize the questions that the persistence answers when applied to real valued maps, and then develop a notion of persistence for circle valued maps which can answer similar questions and more. We fix a field κ and write $H_r(X)$ to denote the homology vector space of X in dimension r with coefficients in a field κ.

Sublevel persistence. The persistent homology introduced in [11] and further developed in [16] is concerned with the following questions:

Q1. Does the class $x \in H_r(X_{(-\infty,t]})$ originate in $H_r(X_{(-\infty,t'']})$ for $t'' < t$? Does the class $x \in H_r(X_{(-\infty,t]})$ vanish in $H_r(X_{(-\infty,t']})$ for $t < t'$?

Q2. What are the smallest t' and t'' such that this happens?

This information is contained in the linear maps $H_r(X_{(-\infty,t]}) \to H_r(X_{(-\infty,t']})$ where $t' \geq t$ and is known as persistence. Since the involved subspaces are sublevel sets, we refer this persistence as *sublevel persistence*. When f is tame, the persistence for each $r = 0, 1, \cdots \dim X$, is determined by a finite collection of invariants referred to as **bar codes** [16]. For sublevel persistence the bar codes are a collection of *closed intervals* of the form $[s, s']$ or $[s, \infty)$ with s, s' being the critical values of f. From these bar codes one can derive the Betti numbers of $X_{(-\infty,a]}$, the dimension of $\mathrm{img}(\mathrm{H_r}(\mathrm{X}_{(-\infty,t]}) \to \mathrm{H_r}(\mathrm{X}_{(-\infty,t']}))$ and get the answers to questions Q1 and Q2. For example, the number of r-bar codes which contain the interval $[a, b]$ is the dimension of $\mathrm{img}(\mathrm{H_r}(\mathrm{X}_{(-\infty,a]}) \to \mathrm{H_r}(\mathrm{X}_{(-\infty,b]}))$. The number of r-bar codes corresponding to the interval $[a, b]$ is the maximal number of linearly independent homology classes born exactly in $X_{(-\infty,a]}$ but not before which also die exactly in $H_r(X_{-\infty,b]})$ but not before.

Level persistence. Instead of sublevels, if we use levels, we obtain what we call level persistence. The level persistence was first considered in [10] but was completely characterized when the zigzag persistence was introduced in [4]. Level persistence is concerned with the homology of the fibers $H_r(X_t)$ and addresses questions of the following type.

Q1. Does the image of $x \in H_r(X_t)$ vanish in $H_r(X_{[t,t']})$, where $t' > t$ or in $H_r(X_{[t'',t]})$, where $t'' < t$?

Q2. Can x be detected in $H_r(X_{t'})$ where $t' > t$ or in $H_r(X_{t''})$ where $t'' < t$? The precise meaning of detection is explained below.

Q3. What are the smallest t' and t'' for the answers to Q1 and Q2 to be affirmative?

To answer such questions one has to record information about the following maps:

$$H_r(X_t) \to H_r(X_{[t,t']}) \leftarrow H_r(X_{t'}).$$

The **level persistence** is the information provided by this collection of vector spaces and linear maps for all t, t'. We say that $x \in H_r(X_t)$ is dead in $H_r(X_{[t,t']}),\ t' > t$, if its image by $H_r(X_t) \to H_r(X_{[t,t']})$ vanishes. Similarly, x is dead in $H_r(X_{[t'',t]}),\ t'' < t$, if its image by $H_r(X_t) \to H_r(X_{[t'',t]})$ vanishes.

We say that $x \in H_r(X_t)$ is detected in $H_r(X_{t'}),\ t' > t$, if its image in $H_r(X_{[t,t']})$ is contained in the image of $H_r(X_{t'}) \to H_r(X_{[t,t']})$. Similarly, the detection of x can be defined for $t'' < t$ also. In Figure 1, the class consisting of the sum of two circles at level t is not detected on the right, but is detected at all levels on the left up to (but not including) the level t'. In case of a tame map the collection of the vector spaces and linear maps is determined up to coherent isomorphisms by a collection of invariants called *bar codes* which are intervals of the form $[s, s'], (s, s'), (s, s'], [s, s')$ with s, s' critical values. These bar codes are called *invariants* because two tame maps $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ which are fiber wise homotopy equivalent have the same associated bar codes. The open end of an interval signifies the death of a homology class at that end (left or right) whereas a closed end signifies that a homology class cannot be detected beyond this level (left or right). Level persistence provides considerably more information than the sub level persistence. The bar codes of the sub level persistence (for a tame map) can be recovered from the ones of level persistence, cf. [1]. It turns out that the bar codes of the level persistence can be also recovered from the bar codes of the sub level persistence of f and additional maps canonically associated to f as it follows from Appendix D.

In Figure 1, we indicate the bar codes both for sub level and level persistence for some simple map in order to illustrate their differences. The reader can easily derive them by using the method described in Appendix D.

3 Persistence for circle valued maps

Let $f : X \to \mathbb{S}^1$ be a circle valued map. The sublevel persistence for such a map cannot be defined since circularity in values prevents defining sub-levels. Even level persistence cannot be defined as per say since the intervals may repeat over values. To overcome this difficulty we associate the infinite cyclic covering map $\tilde{f} : \tilde{X} \to \mathbb{R}$ for f. It is defined by the commutative diagram at left:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{f}} & \mathbb{R} \\ \psi \downarrow & & \downarrow p \\ X & \xrightarrow{f} & \mathbb{S}^1 \end{array}$$

The map $p : \mathbb{R} \to \mathbb{S}^1$ is the universal covering of the circle (the map which assigns to the number $t \in \mathbb{R}$ the angle $\theta = t(mod\ 2\pi)$ and ψ is the pull back of p by the map f which is an infinite cyclic covering. Notice that $X_\theta = \tilde{X}_t$ if $p(t) = \theta$. If $x \in H_r(X_\theta) = H_r(\tilde{X}_t),\ p(t) = \theta$, the questions Q1, Q2, Q3 for f and X can be formulated in terms of the level persistence for $\tilde{f}$ and $\tilde{X}$.

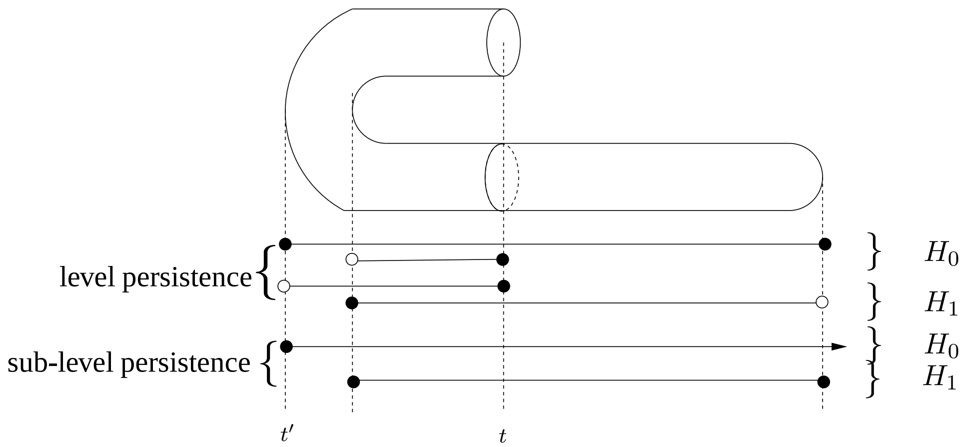


Figure 1: Bar codes for level and sub-level persistence.

Suppose that $x \in H_r(\tilde{X}_t) = H_r(X_\theta)$ is detected in $H_r(\tilde{X}_{t'})$ for some $t' > t$. If the interval $[t, t']$ contains a point t'' so that $p(t'') = \theta$, then, in some sense, x returns to $H_r(X_\theta)$ going along the circle $\mathbb{S}^1$ one or more times. When this happens, the class x may change in some respect . This gives rise to new questions that were not encountered in sublevel or level persistence.

Q4. When $x \in H_r(X_\theta)$ returns, how does the "returned class" compare with the original class x? It may disappear after going along the circle a number of times, or it might never disappear.

To answer Q1-Q4 one has to record information about $H_r(X_\theta) \to H_r(X_{[\theta,\theta']}) \leftarrow H_r(X_{\theta'})$ for any pair of angles θ and θ' which differ by at most 2π. This information is referred to as the **persistence for the circle valued map** f.

When f is tame, this is again completely determined up to coherent isomorphisms by a finite collection of invariants. However, unlike sublevel and level persistence, the invariants include structures other than bar codes called *Jordan cells*. Specifically, for any $r = 0, 1, \cdots, (\dim(M) - 1)$ we have two types of invariants:

- *bar codes*: intervals of type $\{s, s'\}$, $0 < s \leq 2\pi,\ s \leq s' < \infty$, that are closed or open at s or s', precisely of the form $[s, s'], (s, s'], [s, s'), (s, s')$. These intervals can be geometerized as "spirals" with equations 1. For any interval $\{s, s'\}$ the spiral is the plane curve (see Figure 3 in section 4)

$$\begin{aligned} x(\theta) &= \left(\frac{1}{s'-s}\theta + \frac{s'-2s}{s'-s} + 1\right) cos\theta \\ y(\theta) &= \left(\frac{1}{s'-s}\theta + \frac{s'-2s}{s'-s} + 1\right) sin\theta \\ &\theta \in \{s, s'\}. \end{aligned} \tag{1}$$

- *Jordan cells*. A Jordan cell is a pair (λ, k), $\lambda \in \overline{\kappa} \setminus 0,\ \ k \in \mathbb{Z}_{>0}$, where $\overline{\kappa}$ denotes the algebraic closure of the field κ. It corresponds to a $k \times k$ matrix of the form

$$\begin{pmatrix} \lambda & 1 & 0 \ldots & 0 \\ 0 & \lambda & 1 \ldots & 0 \\ \vdots & & & \\ 0 & \ldots & \lambda & 1 \\ 0 & \ldots & 0 & \lambda \end{pmatrix}. \tag{2}$$

Notice that the bar codes for f can be inferred from $\tilde{f} : \tilde{X}_{[a,b]} \to \mathbb{R}$ with $[a, b]$ being any large enough interval. Specifically, the bar codes of $f : X \to \mathbb{S}^1$ are among the ones of $\tilde{f} : \tilde{X}_{[a,b]} \to \mathbb{R}$ for $(b - a)$ large enough. We explain in section 5 how to get a reasonable upper bound on $b - a$. However, the Jordan cells can not be derived from $\tilde{f} : \tilde{X} \to \mathbb{R}$ or any of its truncations $\tilde{f} : \tilde{X}_{[a,b]} \to \mathbb{R}$. The collection of bar codes and Jordan cells for each $r = 1, 2, \cdots (\dim X - 1)$ constitute the r-**invariants** of the map f which we define precisely in the next section.

The end points of any bar code correspond to critical angles, that is, s and $s'(mod\ 2\pi)$ of a bar code interval $\{s, s'\}$ are critical angles. One can recover the following information from the bar codes and Jordan cells:

1. The Betti numbers of each fiber,

2. The Betti numbers of the source space X, and

3. Betti numbers of $\tilde{X}_{[a,b]}$.

Theorems 3.1 and 3.2 make the above statement precise. Let B be a bar code described by a spiral (eqn. 1) and θ be any angle. Let $n_\theta(B)$ denote the cardinality of the intersection of the spiral with the ray originating at the origin and making an angle θ with the x-axis. For the Jordan cell $J = (\lambda, k)$, let $n(J) = k$. Furthermore, let $\mathcal{B}_r$ and $\mathcal{J}_r$ denote the set of bar codes and Jordan cells for H_r-homology. We have the following results.

Theorem 3.1 $\dim H_r(X_\theta) = \sum_{B \in \mathcal{B}_r} n_\theta(B) + \sum_{J \in \mathcal{J}_r} n(J)$.

Theorem 3.2 $\dim H_r(X) = \sharp\{B \in \mathcal{B}_r | \text{both ends closed}\} + \sharp\{B \in \mathcal{B}_{r-1} | \text{both ends open}\} + \sharp\{J \in \mathcal{J}_r | \lambda = 1\} + \sharp\{J \in \mathcal{J}_{r-1} | \lambda = 1\}$.

A real valued tame map $f : X \to \mathbb{R}$ can be regarded as a circle valued tame map $f' : X \to \mathbb{S}^1$ by identifying $\mathbb{R}$ to $(0, 2\pi)$ with critical values $t_1, \cdots, t_m$ becoming the critical angles $\theta_1, \cdots, \theta_m$ where $\theta_i = 2 \arctan t_i + \pi$. The map f' in this case will not have any Jordan cells. We have the following corollary:

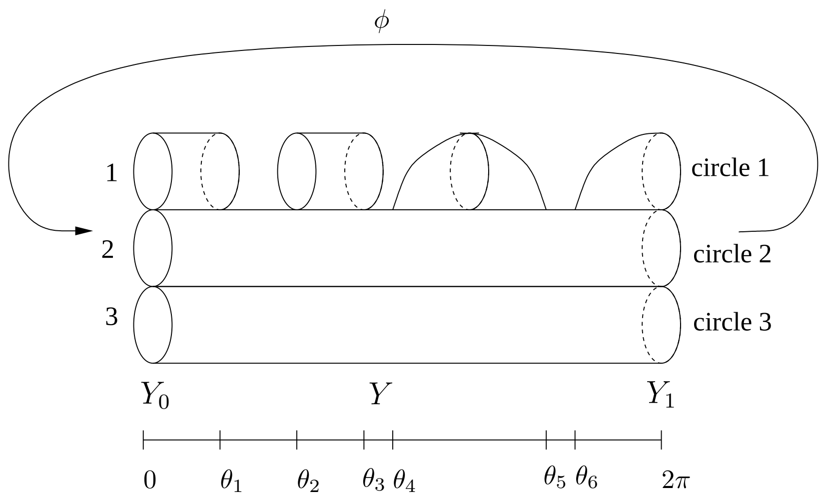
Corollary 3.3 $\dim H_r(X_\theta) = \sum_{B \in \mathcal{B}_r} n_\theta(B)$ *and*
$\dim H_r(X) = \sharp\{B \in \mathcal{B}_r | \text{both ends closed}\} + \sharp\{\mathrm{B} \in \mathcal{B}_{\mathrm{r}-1} | \text{both ends open}\}$.

Theorem 3.1 is quite intuitive and is in analogy with the derived results for sublevel and level persistence [4, 16]. Theorem 3.2 is more subtle. Its counterpart for real valued function (Corollary 3.3) has not yet appeared in the literature. The proofs of these results require the definition of the bar codes and Jordan cells which appear in the next section. The proofs are sketched in Appendix A.

The Questions Q1-Q3 can be answered using the bar codes. The question Q4 about returned homology can be answered using the bar codes and Jordan cells.

Figure 2 indicates a tame map $f : X \to \mathbb{S}^1$ and the corresponding invariants, bar codes, and Jordan cells. The space X is obtained from Y in the figure by identifying its right end Y_1 (a union of three circles) to the left end Y_0 (again a union of three circles) following the map $\phi : Y_1 \to Y_0$. The map $f : X \to \mathbb{S}^1$ is induced by the projection of Y on the interval $[0, 2\pi]$. We have $H_1(Y_1) = H_1(Y_0) = \kappa \oplus \kappa \oplus \kappa$ and ϕ induces a linear map in H_1-homology represented by the matrix

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{pmatrix}.$$



map ϕ	r-invariants		
	dimension	bar codes	Jordan cells
circle 1: 3 times around circle 1 circle 2: 1 time around 2 and 3 times around 3 circle 3: 1 time around 2	0		$(1,1)$
	1	$(\theta_6, \theta_1 + 2\pi]$ $[\theta_2, \theta_3]$ (θ_4, θ_5)	$(1,2)$

Figure 2: Example of r-invariants for a circle valued map

The first generator of $H_1(\tilde{X}_{2\pi})$ given by circle 1 is dead in $H_1(\tilde{X}_{[\theta,2\pi]})$ for $\theta = \theta_6$ but not for $\theta \in (\theta_6, 2\pi]$ and is detected in $H_1(\tilde{X}_{2\pi+\theta})$ for $\theta = \theta_1$ but not for $\theta > \theta_1$. It generates a bar code $(\theta_6, 2\pi + \theta_1]$. The other two generators given by circles 2 and 3 never die; they return with an isomorphism represented by

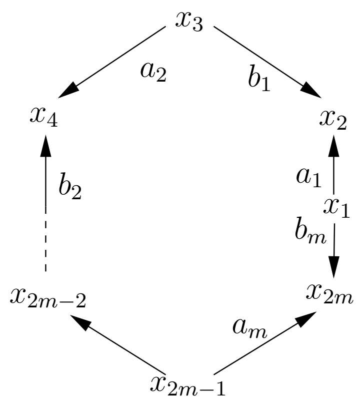
$$\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \text{ whose } \textit{Jordan} \text{ canonical form (see [7]) is the matrix } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which provides a Jordan cell $(1, 2)$.

4 Representation theory and r-invariants

The invariants for the circle valued map are derived from the representation theory of quivers. The quivers are directed graphs. The representation theory of simple quivers such as paths with directed edges was described by Gabriel [8] and is at the heart of the derivation of the invariants for zigzag and then level persistence in [4]. For circle valued maps, one needs representation theory for circle graphs with directed edges. This theory appears in the work of Nazarova [13], and Donovan and Ruth-Freislich [9]. The reader can find a refined treatment in Kac [14].

Let G_{2m} be a directed graph with $2m$ vertices, $x_1, x_1, \cdots x_{2m}$. Its underlying undirected graph is a simple cycle. The directed edges in G_{2m} are of two types: *forward* $a_i : x_{2i-1} \to x_{2i}$, $1 \leq i \leq m$, and *backward* $b_i : x_{2i+1} \to x_{2i}$, $1 \leq i \leq m-1$, $b_m : x_1 \to x_{2m}$.



We think of this graph as being residing on the unit circle centered at the origin o in the plane.

A representation ρ on G_{2m} is an assignment of a vector space V_x to each vertex x and a linear map $V_e : V_x \to V_y$ for each oriented edge $e = \{x, y\}$. Two representations ρ and ρ' are isomorphic if for each vertex x there exists an isomorphism from the vector space V_x of ρ to the vector space V'_x of ρ', and these isomorphisms intertwine the linear maps $V_x \to V_y$ and $V'_x \to V'_y$. A *non-trivial* representation assigns at least one vector space which is not zero-dimensional. A representation is *indecomposable* if it is not isomorphic to the sum of two nontrivial representations. It is not hard to observe that each representation has a decomposition as a sum of indecomposable representations unique up to isomorphisms.

We provide a description of indecomposable representations of the quiver G_{2m}. For any triple of integers $\{i, j, k\}$, $1 \leq i, j \leq m$, $k \geq 0$, one may have any of the four representations, $\rho^I([i, j]; k)$, $\rho^I((i, j]; k)$, $\rho^I([i, j); k)$, and $\rho^I((i, j); k)$ defined below.

Suppose that the evenly indexed vertices $\{x_2, x_4, \cdots x_{2m}\}$ of G_{2m} which are the targets of the directed arrows correspond to the angles $0 < s_1 < s_2 < \cdots < s_m \leq 2\pi$. Draw the spiral curve given by equation (1) for the interval $\{s_i, s_j + 2k\pi\}$; refer to Figure 3.

For each x_i, let $\{e_i^1, e_i^2, \cdots\}$ denote the ordered set (possibly empty) of intersection points of the ray ox_i with the spiral. While considering these intersections, it is important to realize that the point $(x(s_i), y(s_i))$ (resp. $(x(s_j + 2k\pi), y(s_j + 2k\pi))$) does not belong to the spiral if the interval in $\rho^I(\{i, j\}, k)$ is open at i (resp. j). For example, in Figure 3, the last circle on the ray ox_{2j} is not on the spiral since $\rho^I([i, j); 2)$ is open at right.

Let V_{x_i} denote the vector space generated by the base $\{e_i^1, e_i^2, \cdots\}$. Furthermore, let $\alpha_i : V_{x_{2i-1}} \to V_{x_{2i}}$ and $\beta_i : V_{x_{2i+1}} \to V_{x_{2i}}$ be the linear maps defined on bases and extended by linearity as follows: assign to $e^\ell_{2i\pm1}$ the vector $e^h_{2i} \in V_{x_i}$ if e^h_{2i} is an adjacent intersection point to the points $e^\ell_{2i\pm1}$ on the spiral. If e^h_{2i} does not exist, to $e^\ell_{2i\pm1}$ assign zero. If $e^\ell_{2i\pm1}$ do not go to zero, h has to be l, $l-1$, or $l+1$.

The construction above provides a representation on G_{2m} which is indecomposable. One can also think these representations as the bar codes $[s_i, s_j + 2k\pi]$, $(s_i, s_j + 2k\pi]$, $[s_i, s_j + 2k\pi)$, and $(s_i, s_j + 2k\pi)$.

For any Jordan cell (λ, k) we associate a representation $\rho^J(\lambda, k)$ defined as follows. Assign the vector space with base $e_1, e_2, \cdots, e_k$ to each x_i and take all linear maps α_i but one (say α_1) and β_i the identity. The map α_1 is given by the Jordan cell matrix (λ, k). Again this representation is indecomposable.

It follows from the work of [9, 13, 14] that bar codes and Jordan cells as constructed above are all and only indecomposable representations of the quiver G_{2m}.

Observation 4.1 *If a representation ρ does not contain any indecomposable representations of type ρ^I in its decomposition, all linear maps α_is and β_is are isomorphisms. For such a representation, starting with an index i, consider the linear isomorphism*

$$T_i = \beta_i^{-1} \cdot \alpha_i \cdot \beta_{i-1}^{-1} \cdot \alpha_{i-1} \cdots \beta_2^{-1} \cdot \alpha_2 \cdot \beta_1^{-1} \cdot \alpha_1 \cdot \beta_m^{-1} \cdot \alpha_m \cdot \beta_{m-1}^{-1} \cdot \alpha_{m-1} \cdots \beta_{i+1}^{-1} \cdot \alpha_{i+1}.$$

The Jordan canonical form of the isomorphism T_i is independent of i and is a block diagonal matrix with the diagonal consisting of Jordan cells (λ, k)s. Clearly, ρ is the direct sum of $\rho^J(\lambda, k)$s with each (λ, k) being a Jordan cell of T_i.

So, for a general representation ρ, once we separate out all indecomposables of type ρ^I, we can obtain the ones corresponding to the Jordan cell from the Jordan canonical form of the linear isomorphism T_i. A procedure to decompose a representation ρ as a sum of indecomposable is provided in Appendix E.

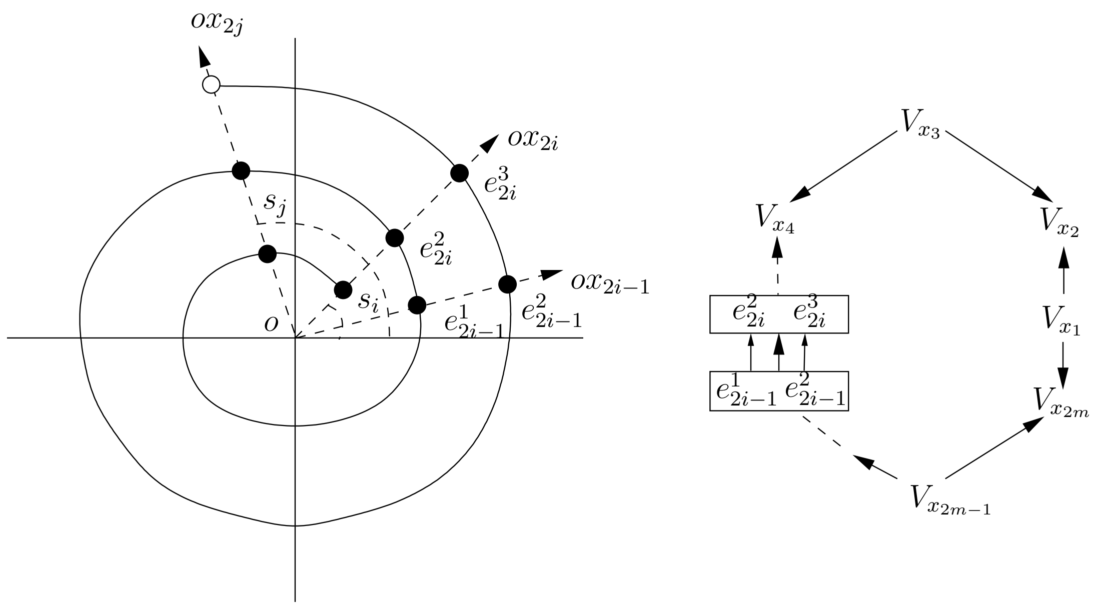


Figure 3: The spiral for $[s_i, s_j + 4\pi)$.

The r-invariants. Let f be a circle valued tame map defined on a topological space X. For f with m critical angles $0 < s_1 < s_2, \cdots s_m \leq 2\pi$, consider the quiver G_{2m} with the vertices x_{2i} specified by the angles s_i and the vertices x_{2i-1} by the angles t_i that satisfy $0 < t_1 < s_1 < t_2 < s_2, \cdots t_m < s_m$.

For any r, consider the representation ρ_r of G_{2m} with $V_{x_i} = H_r(X_{x_i})$ and the linear maps α_is and β_is induced in the H_r-homology by the maps $a_i : X_{x_{2i-1}} \to X_{x_{2i}}$ and $b_i : X_{x_{2i+1}} \to X_{x_{2i}}$ described in section 2. The bar codes and the Jordan cells of this representation are independent of the choice of t_is and are the r**-invariants** of f.

5 Algorithm

Given a circle valued tame map $f : X \to \mathbb{S}^1$, we now present an algorithm to compute the bar codes when X is a finite simplicial complex, and f is generic and linear. Genericity means that f is injective on vertices. The genericity condition is not necessary if one starts with an angle θ which is not the image of any vertex. To explain linearity we recall that, for any simplex $\sigma \in X$, the restriction $f|_\sigma$ admits liftings $\hat{f} : \sigma \to \mathbb{R}$, i.e. $\hat{f}$ is a continuous map which satisfies $p \cdot \hat{f} = f|_\sigma$. The map $f : X \to \mathbb{S}^1$ is called *linear* if for any simplex σ, at least one of the liftings (and then any other) is linear.

Most of tame circle valued maps of practical interest are fiber wise equivalent to linear circle valued maps defined on a simplicial complex.

Given a tame circle valued map $f : X \to \mathbb{S}^1$, we consider the infinite cyclic covering map $\tilde{f} : \tilde{X} \to \mathbb{R}$ of $f : X \to \mathbb{S}^1$ as described in section 4. Precisely, one has the infinite cyclic coverings $\psi : \tilde{X} \to X$, and $p : \mathbb{R} \to \mathbb{S}^1$ with $p(t) = t(mod\, 2\pi)$, which satisfies $p \cdot \tilde{f} = f \cdot \psi$. Recall that $p(t) = \theta$ implies $\tilde{X}_t = X_\theta$.

Since $\tilde{X}$ is not compact, we truncate $\tilde{X}$ to $\tilde{X}_{[t,t+2\pi k]}$. The restriction $\tilde{f} : \tilde{X}_{[t,t+2\pi k]} \to \mathbb{R}$ is tame. In view of Theorem 3.1, if k is an integer larger than $(\inf_\theta \dim H_r(X_\theta)) + 1$, the r-bar codes of f are among the (level persistence) r-bar codes of $\tilde{f} : \tilde{X}_{[t,t+2\pi k]} \to \mathbb{R}$. One can verify that if $t \in [0, 2\pi]$, they are the same as the r-bar codes which begin in the interval $(2\pi, 4\pi]$ for the truncation $\tilde{f} : \tilde{X}_{[t,t+2\pi k]} \to \mathbb{R}$. To compute the r-bar codes for f we proceed as follows.

Step 1. Choose a $\theta \in [0, 2\pi]$, compute the rank d of $H_r(X_\theta)$, and take $k = d+2$. Since X_θ is a cell complex we can represent its incidence structure with a matrix. One may apply the standard persistence algorithm to this matrix for computing the rank of $H_r(X_\theta)$. Computing the cell complex X_θ from X explicitly is cumbersome. We propose a simple method to compute the incidence matrix of X_θ from that of X. Let $M(C)$ denote the incidence matrix of any finite cell complex C. For any θ, consider a matrix $\hat{M}$ as the minor of $M(X)$ consisting of the rows and columns i with $\theta \in f(\text{int}\,\sigma_\text{i})$ and regard the simplex σ_i as the cell $\hat{\sigma}_i$ with $\dim(\hat{\sigma}_i) = \dim(\sigma_i) - 1$. If there are $\ell < n$ such simplices, $\hat{M}$ is an $\ell \times \ell$ matrix.

- If no vertex takes the value θ, then $M(X_\theta) = \hat{M}$ is the incidence matrix of the cell complex X_θ.
- If there exists a vertex v (unique since f is generic) so that $f(v) = \theta$, then the incidence matrix $M(X_\theta)$ is an $(\ell+1) \times (\ell+1)$ matrix with one additional row and column corresponding to the vertex v viewed as a cell of dimension 0 which should be indexed before all other cells. The entry for the pair $(v, \hat{\sigma}_j)$ in $M(X_\theta) = 1$ if σ_j is a $2-$simplex which has v as a vertex and 0 otherwise.

The second situation can be avoided by choosing θ different from the value of f on vertices.

Step 2. After determining k in the previous step, we construct $\tilde{X}_{[t,t+2\pi k]}$ by first cutting open X at $X_{\theta=t}$ for some $t \in [0, 2\pi]$ and then putting copies of this dissected X one after another joining along copies of X_θ. See Figure 4. It turns out that we do not need to compute explicitly the space $\tilde{X}_{[t,t+2\pi k]}$, but instead can compute its incidence matrix from the matrix $M(X)$. For a more formal description of the process, see Appendix C.

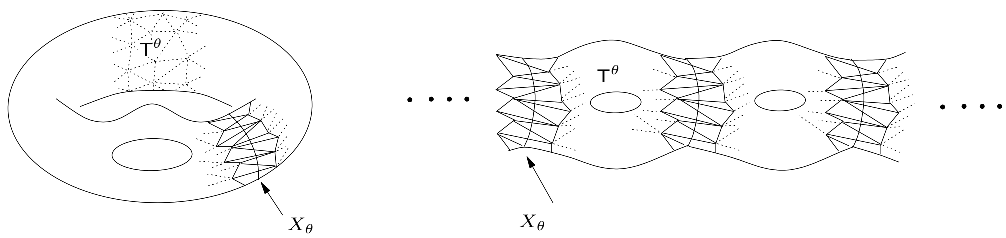


Figure 4: Complex X with level X_θ on left. Part of the complex $\tilde{X}$ on right.

Step 3. The algorithm for zigzag persistence [4] can compute the bar codes for a real-valued function defined on a finite simplicial complex. Once we compute the incidence matrix of $\tilde{X}_{[t,t+2\pi k]}$ that is compatible with $\tilde{f}$, we can apply the zigzag persistence algorithm to compute the bar codes. In Appendix D we provide an alternative algorithm to compute the bar codes using the standard persistence algorithm [11, 16].

The computation of Jordan cells is described in Appendix E where both bar codes and Jordan cells are computed, and in Appendix F where only Jordan cells are computed. Both methods assume that the quiver representation ρ_r is available. More precisely, the quiver representation ρ_r defined by $f : X \to \mathbb{S}^1$ in H_r-homology is described by matrices that represent maps α_is and β_is. These matrices can be computed by standard persistence algorithm. The Jordan cells and the bar codes can be derived from these matrices following the procedures in Appendix E and Appendix F. These procedures require matrix manipulations which can be implemented with standard algorithms in linear algebra.

6 1-cocycles

In this section we discuss how one may leverage the definition and computation of persistence for circle valued maps to derive the same for 1-cocycles. These 1-cocycles appear in many ways in science. For example a smooth curl free vector field on a Riemannian manifold defines 1-cocycle for any triangulation of the manifold by integration along the edges. Similarly a ranking problem which is locally consistent [18, 19] defines such a 1-cocycle. The persistence invariants that we define for such 1-cocycles carry information about the dynamics of the vector field in the first case (at least as much as Novikov theory provides [3]) and the quality of the global inconsistency of the ranking in the second.

Let X be a simplicial complex with $\mathcal{X}_0$ being the set of vertices. Denote by $\mathcal{X}_1 \subseteq \mathcal{X}_0 \times \mathcal{X}_0$ the collection of pairs (x, y) with $x, y \in \mathcal{X}_0$ so that x, y are the end points of a 1-simplex in X. Note that if $(x, y) \in \mathcal{X}_1$ then $(y, x) \in \mathcal{X}_1$.

A 0-*cochain* is a function $f : \mathcal{X}_0 \to \mathbb{R}$. A 0-cochain can be identified with a continuous map $f : X \to \mathbb{R}$ whose restriction to each simplex is linear. The 0-cochain f is *generic* if $f : \mathcal{X}_0 \to \mathbb{R}$ is injective. The 1-cochains are a generalization of 0-cochains in that their domain is the set $\mathcal{X}_1$ of oriented edges of X. Let $\mathbf{f} : \mathcal{X}_1 \to \mathbb{R}$ be a 1-cochain defined on X. The map $\mathbf{f}$ is a 1-*cocycle* if it satisfies:

1. $\mathbf{f}(x, y) = -\mathbf{f}(y, x)$ for any ordered pair $(x, y) \in \mathcal{X}_1$, and
2. if $(x, y, z) \in \mathcal{X}_2$ then $\mathbf{f}(x, y) + \mathbf{f}(y, z) + \mathbf{f}(z, x) = 0$; equivalently $\mathbf{f}(x, y) + \mathbf{f}(y, z) = \mathbf{f}(x, z)$.

If $St(x)$ denotes the star of the vertex $x \in \mathcal{X}_0$ (the star of any simplex is a sub complex), a 1-cocycle $\mathbf{f}$ defines a unique function $f_x : St(x) \to \mathbb{R}$ by the formulae $f_x(x) = 0$ and $f_x(y) = \mathbf{f}(x, y)$ for any vertex $y \neq x$ in $St(x)$. Clearly $(f_x - f_y)(z)$ is constant in z for any z in a connected component of $St(x) \cap St(y)$.

Thus, a 1-cocycle can be thought of as a collection of linear maps $\{f_x : St(x) \to \mathbb{R}\}$ for each vertex x, such that the difference $f_x - f_y$ is constant on each connected component of $St(x) \cap St(y)$. A 1-cocycle $\mathbf{f}$ is *generic* if all linear maps f_x are generic, i.e., injective when restricted to vertices of $St(x)$.

Any 1-cocycle $\mathbf{f}$ represents a cohomology class $< \mathbf{f} > \in H^1(X; \mathbb{R})$ and any such cohomology class is represented by a 1-cocycle. Two 1-cocycles $\mathbf{f}_1$ and $\mathbf{f}_2$ represent the same cohomology class iff $\mathbf{f}_1 - \mathbf{f}_2 = \delta f$ for some 0-cochain f.

An *almost integral* 1-cocycle is a pair $(\mathbf{f}, \alpha)$ where $\mathbf{f}$ is a 1-cocycle whose values on integral 1-cycles are integer multiple of a fixed positive real α.[2] These cocycles include the class of *rational* 1-cocycles whose values on integral 1-cycles are rational numbers. In particular, a 1-cocycle $\mathbf{f} : \mathcal{X}_1 \to \mathbb{R}$ with rational numbers as values is a rational 1-cocycle and therefore an almost integral 1-cocycle for some rational number α. One can show that an almost integral 1-cocycle corresponds to a circle valued map and vice versa.

Proposition 6.1 *Any circle valued map $f : X \to \mathbb{S}^1$ defines an almost integral 1-cocycle $(\mathbf{f}, \alpha)$ and any almost integral 1-cocycle $(\mathbf{f}, \alpha)$ defines a circle valued map $f : X \to \mathbb{S}^1 = \mathbb{R}/\alpha$ whose associated 1-cocycle is $(\mathbf{f}, \alpha)$.*

A proof is given in Appendix B.

Persistence of an (almost integral) 1-cocycle: We define the persistence of an almost integral 1-cocycle and therefore of any rational 1-cocycle (for α the largest positive rational number which makes the rational 1-cocycle almost integral) as the persistence of the associated circle valued map.

7 Conclusions

We have analyzed circle valued maps from the perspective of topological persistence. We show that the notion of persistence for such maps incorporate an invariant that is not encountered in persistence studied

[2] in the language of algebraic topology the cohomology class $< \mathbf{f} > \in H^1(X; \mathbb{R})$ has degree of rationality 1, or equivalently the image of the induced homomorphism $H_1(X; \mathbb{Z}) \to \mathbb{R}$ is $\alpha\mathbb{Z} \subset \mathbb{R}$.

erstwhile. Our results also shed lights on computing ranks of homological vector spaces from bar codes (Theorems 3.1 and 3.2). We have given algorithms to compute the bar codes of the invariants, one uses the zigzag persistence algorithm and the other uses standard persistence algorithm. We have also proposed an approach to compute the Jordan cells.

Some open questions ensue from this research. What can be said about the stability of the invariants as was established for standard persistence [5]? How can we convert the approaches sketched in Appendices E and F for computing bar codes and Jordan cells into an efficient algorithm? For 1-cocyles, we define and compute persistence via a circle valued map. Is it possible to skip this intermediate map and define invariants directly?

The standard persistence is related to Morse theory. In a similar vein, the persistence for circle valued map is related to Morse Novikov theory [15]. The work of Burghelea and Haller applies Morse Novikov theory to instantons and closed trajectories for vector field with Lyapunov closed one form [3]. The results in this paper will very likely provide additional insight on the dynamics of these vector fields [2].

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Appendix

A Proofs

A careful look at Figure 2 and the bar codes indicate why a semi-closed (open one end closed the other) bar code does not contribute to the homology of the total space X and why a closed r-bar code (both ends closed) contributes one unit while an open (both end open) $(r-1)$-bar code contributes one unit to the r-homology of the total space. The lack of contribution of a Jordan cell with $\lambda \neq 1$ and the contribution of a r-Jordan cell with $\lambda = 1$ of one unit to both r and $r+1$ dimensional homology of the total space can be explained with the homology of mapping torus. Below we will explain rigorously but schematically the arguments for the proof of Theorems 3.1, 3.2 and Corollary 3.3.

First recall that a representation ρ of the graph G_{2m} is indicated by the vector spaces $V_{x_{2i-1}}, V_{x_{2i}}$ and the linear maps α_i and β_i. To such representation ρ we associate the block matrix $M_\rho : \bigoplus_{1\leq i\leq m} V_{x_{2i-1}} \to \bigoplus_{1\leq i\leq m} V_{x_{2i}}$ defined by:

$$\begin{pmatrix} \alpha_1 & -\beta_1 & 0 & \dots & \dots & 0 \\ 0 & \alpha_2 & -\beta_2 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & \dots & \dots & \dots & \dots \alpha_{m-1} & -\beta_{m-1} \\ -\beta_m & \dots & \dots & \dots & \dots & \alpha_m \end{pmatrix}$$

For this representation we define the "dimension" as the $2m$-tuple of positive integers $\dim(\rho) = (n_1, r_1 \cdots n_m, r_m)$ with $n_i = \dim V_{x_{2i-1}}$ and $r_i = \dim V_{x_{2i}}$ and the numbers $\ker(\rho) = \dim \ker M_\rho$ and $coker(\rho) = \dim coker M_\rho$. For the sum of two such representations $\rho = \rho_1 \oplus \rho_2$ we have:

Proposition A.1

1. $\dim(\rho_1 \oplus \rho_2) = \dim(\rho_1) + \dim(\rho_2)$,
2. $\ker(\rho_1 \oplus \rho_2) = \ker(\rho_1) + \ker(\rho_2)$,
3. $coker(\rho_1 \oplus \rho_2) = coker(\rho_1) + coker(\rho_2)$.

The explicit description of the representations corresponding to the bar codes permits explicit calculations.

Proposition A.2

1. *If* $i \leq j$ *then*
   (a) $\dim \rho^I([i,j],k)$ *is given by:*
   $n_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
   $r_l = k+1$ *if* $i \leq l \leq j$ *and* k *otherwise*
   (b) $\dim \rho^I((i,j],k)$ *is given by:*
   $n_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
   $r_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
   (c) $\dim \rho^I([i,j),k)$ *is given by:*
   $n_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
   $r_l = k+1$ *if* $i \leq l \leq (j-1)$ *and* k *otherwise,*

(d) $\dim \rho^I((i,j),k)$ *is given by:*
$n_l = k+1$ *if* $(i+1) \leq l \leq j$ *and* k *otherwise,*
$r_l = k+1$ *if* $(i+1) \leq l \leq (j-1)$ *and* k *otherwise*

2. *If* $i > j$ *then similar statements hold.*

(a) $\dim \rho^I([i,j];k)$ *is given by:*
$n_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise;*
$r_l = k$ *if* $(j+1) \leq l \leq (i-1)j$ *and* $k+1$ *otherwise*

(b) $\dim \rho^I((i,j];k)$ *is given by:*
$n_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise.*
$r_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise,*

(c) $\dim \rho^I([i,j);k)$ *is given by:*
$n_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise;*
$r_l = k$ *if* $j \leq l \leq (i-1)$ *and* $k+1$ *otherwise,*

(d) $\dim \rho^I((i,j),k)$ *is given by:*
$n_l = k$ *if* $(j+1) \leq l \leq i$ *and* $k+1$ *otherwise;*
$r_l = k$ *if* $j \leq l \leq i$ *and* $k+1$ *otherwise.*

3. $\dim \rho^J(\lambda,k)$ *is given by* $n_i = r_i = k$

Proposition A.3

1. $\ker \rho^I([i,j],k) = 0$, $coker \rho^I([i,j],k) = 1$,
2. $\ker \rho^I([i,j),k) = 0$, $coker \rho^I([i,j),k) = 0$,
3. $\ker \rho^I((i,j],k) = 0$, $coker \rho^I((i,j],k) = 0$,
4. $\ker \rho^I((i,j),k) = 1$, $coker \rho^I((i,j),k) = 0$,
5. $\ker \rho^J(\lambda,k) = 0$ (resp. 1) if $\lambda \neq 1$ (resp. 1),
6. $coker \dim \rho^J(\lambda,k) = 0$ (resp. 1) if $\lambda \neq 1$ (resp. 1).

The proof of Theorem 3.1 is a consequence of Propositions A.1 and A.2. The proof of Theorem 3.2 goes on the following lines. First observe that, up to homotopy, the space X can be regarded as the iterated mapping torus $\mathcal{T}$ described below. Consider the collection of spaces and continuous maps:

$$X_m = X_0 \stackrel{\beta_0=\beta_m}{\longleftarrow} R_1 \stackrel{\alpha_1}{\longrightarrow} X_1 \stackrel{\beta_1}{\longleftarrow} R_2 \stackrel{\alpha_2}{\longrightarrow} X_2 \cdots X_{m-1} \stackrel{\beta_{m-1}}{\longleftarrow} R_m \stackrel{\alpha_m}{\longrightarrow} X_m$$

with $R_i := X_{t_i}$ and $X_i := X_{s_i}$ and denote by $\mathcal{T} = T(\alpha_1 \cdots \alpha_m; \beta_1 \cdots \beta_m)$ the space obtained from the disjoint union

$$\Big(\bigsqcup_{1\leq i\leq m} R_i \times [0,1]\Big) \sqcup \Big(\bigsqcup_{1\leq i\leq m} X_i\Big)$$

by identifying $R_i \times \{1\}$ to X_i by α_i and $R_i \times \{0\}$ to X_{i-1} by β_{i-1}. Denote by $f^{\mathcal{T}} : \mathcal{T} \to [0,m]$ where $f^{\mathcal{T}} : R_i \times [0,1] \to [i,i+1]$ is the projection on $[0,1]$ followed by the translation of $[0,1]$ to $[i,i+1]$. This map is a homotopical reconstruction of $f : X \to \mathbb{S}^1$ provided that, with the choice of angles t_i, s_i and maps a_i b_i described in section 2, $X_i := f^{-1}(s_i), R_i := f^{-1}(t_i), \alpha_i := a_i, \beta_i := b_i$.

Let $\mathcal{P}'$ denote the space obtained from the disjoint union

$$(\bigsqcup_{1\le i\le m} R_i \times (\epsilon, 1]) \sqcup (\bigsqcup_{1\le i\le m} X_i)$$

by identifying $R_i \times \{1\}$ to X_i by α_i, and $\mathcal{P}''$ denote the space obtained from the disjoint union

$$(\bigsqcup_{1\le i\le m} R_i \times [0, 1-\epsilon) \sqcup (\bigsqcup_{1\le i\le m} X_i)$$

by identifying $R_i \times \{0\}$ to X_{i-1} by β_{i-1}.

Let $\mathcal{R} = \bigsqcup_{1\le i\le m} R_i$ and $\mathcal{X} = \bigsqcup_{1\le i\le m} X_i$. Then, one has:

1. $\mathcal{T} = \mathcal{P}' \cup \mathcal{P}''$,
2. $\mathcal{P}' \cap \mathcal{P}'' = (\bigsqcup_{1\le i\le m} R_i \times (\epsilon, 1-\epsilon)) \sqcup \mathcal{X}$, and
3. the inclusions $(\bigsqcup_{1\le i\le m} R_i \times \{1/2\}) \sqcup \mathcal{X} \subset \mathcal{P}' \cap \mathcal{P}''$ as well as the obvious inclusions $\mathcal{X} \subset \mathcal{P}'$and $\mathcal{X} \subset \mathcal{P}''$ are homotopy equivalences.

The Mayer-Vietoris long exact sequence leads to the diagram

$$\begin{array}{ccccccccc}
 & & H_r(\mathcal{R}) & \xrightarrow{M_r(\alpha,\beta)} & H_r(\mathcal{X}) & & \\
 & \nearrow & \uparrow{\scriptstyle pr_1} & & \uparrow{\scriptstyle (Id,-Id)} & \searrow & \\
\cdots \longrightarrow H_{r+1}(\mathcal{T}) & \xrightarrow{\partial_{r+1}} & H_r(\mathcal{R}) \oplus H_r(\mathcal{X}) & \xrightarrow{N} & H_r(\mathcal{X}) \oplus H_r(\mathcal{X}) & \xrightarrow{(i^r,-i^r)} & H_r(\mathcal{T}) \longrightarrow \\
 & & \uparrow{\scriptstyle in_2} & & \uparrow{\scriptstyle \Delta} & & \\
 & & H_r(\mathcal{X}) & \xrightarrow{Id} & H_r(\mathcal{X}) & &
\end{array}$$

Here Δ denotes the diagonal, in_2 the inclusion on the second component, pr_1 the projection on the first component, i^r the linear map induced in homology by the inclusion $\mathcal{X} \subset \mathcal{T}$, and $M_r(\alpha, \beta)$ the map given by matrix

$$\begin{pmatrix}
\alpha_1^r & -\beta_1^r & 0 & \ldots & \ldots & 0 \\
0 & \alpha_2^r & -\beta_2^r & \ldots & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & \ldots & \ldots & \ldots & \ldots \alpha_{m-1}^r & -\beta_{m-1}^r \\
-\beta_m^r & \ldots & \ldots & \ldots & \ldots & \alpha_m^r
\end{pmatrix}$$

with $\alpha_i^r : H_r(R_i) \to H_r(X_i)$ and $\beta_i^r : H_r(R_{i+1}) \to H_r(X_i)$ induced by the maps α_i and β_i, and N defined by

$$\begin{pmatrix} \alpha^r & Id \\ -\beta^r & Id \end{pmatrix}$$

where α^r and β^r are the matrices

$$\begin{pmatrix}
\alpha_1^r & 0 & \ldots & \ldots & 0 \\
0 & \alpha_2^r & \ldots & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & \alpha_{m-1}^r
\end{pmatrix}$$

$$\begin{pmatrix} 0 & \beta_1^r & 0 & \dots & 0 \\ 0 & 0 & \beta_2^r & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & \beta_{m-1}^r \\ \beta_m^r & 0 & \dots & 0 & 0 \end{pmatrix}.$$

As a consequence we obtain

$$H_r(\mathcal{T}) = \mathrm{coker} \mathrm{M_r}(\alpha, \beta) \oplus \ker \mathrm{M_{r-1}}(\alpha, \beta) \tag{3}$$

Theorem 3.2 follows from Propositions A.1, A.3 and the equation 3 above.

B Proof of Proposition 6.1

From circle valued map to 1-cocycle. Consider a continuous *circle valued* map $f : X \to \mathbb{S}^1$. Let $p : \mathbb{R} \to \mathbb{S}^1 = \mathbb{R}/\alpha\mathbb{Z}$ be the map defined by $p(t) = t(\mathrm{mod}\ \alpha)$, α a positive real number. For any simplex $\sigma \in X$, the restriction $f|_\sigma$ admits liftings $\hat{f} : \sigma \to \mathbb{R}$, i.e., $\hat{f}$ is a continuous map which satisfies $p \cdot \hat{f} = f|_\sigma$. Assign to each pair $(x, y) \in \mathcal{X}_1$, $\mathbf{f}(x, y) = \hat{f}(y) - \hat{f}(x)$ where $\hat{f}$ is a lift of f. The assignment is independent of the lifting $\hat{f}$. We obtain an almost integral 1-cocycle $(\mathbf{f}, \alpha)$.

Construction of the covering $\psi : \tilde{X} \to X$. Regard X as a topological space. Choose a base point $x \in X$ and consider the space of continuous paths $\gamma : [0, 1] \to X$ with $\gamma(0) = x$, equipped with compact open topology. Make two continuous paths γ_1 and γ_2 equivalent iff the $\gamma_1(1) = \gamma_2(1)$ and the closed path $\gamma_1 \star \gamma_2^{-1}$ satisfies $< \mathbf{f} > ([\gamma_1 \star \gamma^{-1}]) = 0$. Here $\star$ denotes the concatenation of the paths γ_1 and γ_2^{-1} defined by $\gamma_2^{-1}(t) = \gamma_2(1-t)$, and $[\gamma_1 \star \gamma_2^{-1}]$ denotes the homology class of $\gamma_1 \star \gamma_2^{-1}$. The quotient space $\tilde{X}$, whose underlying set is the set of equivalence classes of paths, is equipped with the canonical map $\psi : \tilde{X} \to X$ induced by assigning to γ the point $\gamma(1) \in X$. The map ψ is a local homeomorphism and $\tilde{X}$ is the total space of a principal covering with group $G = \mathrm{img}(< \mathbf{f} >: H_1(X; \mathbb{Z}) \to \mathbb{R})$, which when $\mathbf{f}$ is almost integral, is isomorphic to $\mathbb{Z}$. If X is equipped with a triangulation, then $\tilde{X}$ gets a triangulation whose simplices, when viewed as subsets of $\tilde{X}$, are homeomorphic by ψ to simplices of X, when viewed as subsets of X.

Construction of $\tilde{f}$ from an almost integral 1-cocycle $(\mathbf{f}, \alpha)$.

Step 1. Consider $\psi : \tilde{X} \to X$ the principal $\mathbb{Z}-$covering associated with the cohomology class $< \mathbf{f} >$ defined by $\mathbf{f}$. This means that $\tilde{X}$ is a simplicial complex equipped with a free simplicial action $\mu : \mathbb{Z} \times \tilde{X} \to \tilde{X}$ whose quotient space, $\tilde{X}/\mu$, is the simplicial complex X. The map ψ identifies to $\tilde{X} \to \tilde{X}/\mu = X$ and satisfies $\psi^*(< \mathbf{f} >) = 0$.

Choose a vertex x of X and call it a *base point*. Notice that the vertices $\tilde{\mathcal{X}}_0$ of $\tilde{X}$ can be also described as equivalence classes of sequences $\{x = x_0, x_1, \cdots x_{N-1}, x_N\}$ with x_is being consecutive vertices of X (i.e. x_i, x_{i+1} are vertices of an edge). Two such sequences, $\{x = x_0, x_1, \cdots, x_{N-1}, x_N\}$ and $\{x = y_0, y_1, \cdots y_{L-1}, y_L\}$ are equivalent if $x_N = y_L$ and the sequence $\{x = z_0, \cdots, z_{N+L} = x\}$ with $z_i = x_i$ if $i \leq N$ and $z_{j+N} = y_{L-j}$ if $j \leq L$, satisfies

$$\sum_{0 \leq i \leq L+N-1} \mathbf{f}(z_i, z_{i+1}) = 0.$$

Step 2. Define the map $\tilde{f} : \tilde{\mathcal{X}}_0 \to \mathbb{R}$ by $\tilde{f}(\tilde{y}) := \sum_{0 \le i \le L-1} \mathbf{f}(y_i, y_{i+1})$ where $\tilde{y} \in \tilde{\mathcal{X}}_0$ is the vertex corresponding to the equivalent class of $\{x = y_0, \cdots, y_L\}$. The description of $\tilde{X}$ given above guarantees that $\tilde{f}$ is well defined. Extend $\tilde{f}$ to a linear map $\tilde{f} : \tilde{X} \to \mathbb{R}$. Observe that if $\tilde{y}_1$ and $\tilde{y}_2$ satisfy $\psi(\tilde{y}_1) = \psi(\tilde{y}_2)$ then $\tilde{f}(\tilde{y}_1) - \tilde{f}(\tilde{y}_2) \in \alpha\mathbb{Z}$. In addition, if $\tilde{e}_1$ and $\tilde{e}_2$ are two edges of $\tilde{X}$ from $\tilde{y}_1$ to $\tilde{y}'_1$ and $\tilde{y}_2$ to $\tilde{y}'_2$ respectively with $\psi(\tilde{e}_1) = \psi(\tilde{e}_2)$, then $\tilde{f}(\tilde{y}'_1) - \tilde{f}(\tilde{y}'_2) = \tilde{f}(\tilde{y}_1) - \tilde{f}(\tilde{y}_2)$. This implies that if $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ are two simplices with $\psi(\tilde{\sigma}_1) = \psi(\tilde{\sigma}_2) = \sigma$ and ψ'_is are the restrictions of ψ to $\tilde{\sigma}_i$ (ψ are bijections on their images), then $\tilde{f} \cdot \psi_1^{-1} - \tilde{f} \cdot \psi_2^{-1} : \sigma \to \mathbb{R}$, is constant and this constant is an integer multiple of the fixed real number α.

From cocycles to circle valued maps. Assume that $(\mathbf{f}, \alpha)$, an almost integral 1-cocycle, has been given. Observe that the map $p \cdot \tilde{f} : \tilde{X} \to \mathbb{R}$ (with $p : \mathbb{R} \to \mathbb{S}^1 = \mathbb{R}/\alpha\mathbb{Z}$) factors through $\tilde{X}/\mu = X$ inducing a circle valued map from X to $\mathbb{S}^1$. This is our circle valued map whose associated 1-cocycle is $\mathbf{f}$.

C Construction of $\tilde{X}_{[t,t+2\pi k]}$

We show how to construct a simplicial complex from X that contains $\tilde{X}_{[t,t+2\pi k]}$ Before we describe the construction, we need some properties of ordering of simplices which we apply later.

Suppose that X is any complex equipped with a filtration $\mathcal{F} = X_0 \subseteq \cdots X_i \subseteq X_{i+1} \cdots \subseteq X_m = X$ with X_is being a subcomplex of X. Let $\mathcal{X}$ denote the set of simplices in X.

Definition C.1 *A total order on $\mathcal{X} = \{\sigma_1, \ldots, \sigma_n\}$ is called topologically consistent if the condition A below is satisfied and filtration compatible if the condition B below is satisfied.*

- **Condition A.** σ_i *is a face of* σ_j *implies* $i < j$.
- **Condition B.** $\sigma_i \in X_k$ *and* $\sigma_j \in X_{k'} \setminus X_k$ *with* $k < k'$ *implies* $i < j$.

Given a filtration $\mathcal{F}$, one can canonically modify any total order which satisfies Condition A into one which satisfies both conditions A and B.

Now consider the input complex X on which the circle valued map f is defined. For any $\theta \in \mathbb{S}^1$, decompose $\mathcal{X}$ as a disjoint union $\mathcal{X} = \mathcal{T}^\theta \sqcup \mathcal{L}^\theta \sqcup \partial_- \mathcal{L}^\theta \sqcup \partial_+ \mathcal{L}^\theta$ where (see Figure 5)

- $\mathcal{L}^\theta$ consists of the set of all simplices whose closure do intersect the level X_θ. Let L^θ be the simplicial complex generated by simplices in $\mathcal{L}^\theta$,
- $\mathcal{T}^\theta$ is the set of simplices which do not belong to L^θ. Let T^θ denote the simplicial complex generated by the the simplices in $\mathcal{T}^\theta$ and consider $\mathsf{T}^\theta \cap \mathsf{L}^\theta$. This simplicial complex is the disjoint union of two simplicial complexes $\partial_- \mathsf{L}^\theta$ and $\partial_+ \mathsf{L}^\theta$ characterized by $f(\sigma) < \theta$ for $\sigma \in \partial_- \mathsf{L}^\theta$ and $f(\sigma) > \theta$ for $\sigma \in \partial_+ \mathsf{L}^\theta$.
- $\partial_\pm \mathcal{L}^\theta$ represent the simplices in $\partial_\pm \mathsf{L}^\theta$.

Our purpose is to build a collection of simplicial complexes which contain $\tilde{X}_{[t,t+2\pi k]}$ where $p(t) = \theta$ and calculate the bar codes contained in $[t, t + 2\pi k]$.

Introduce a nested sequence of simplices $\tilde{\mathcal{X}}^\theta(0) \subseteq \tilde{\mathcal{X}}^\theta(1) \subseteq \cdots \tilde{\mathcal{X}}^\theta(k)$ as follows. Since we will repeat copies of each of the sets $\mathcal{T}^\theta$, $\mathcal{L}^\theta$, $\partial_-\mathcal{L}^\theta$, and $\partial_+\mathcal{L}^\theta$, let $\mathcal{T}^\theta(n)$, $\mathcal{L}^\theta(n)$, $\partial_-\mathcal{L}^\theta(n)$, and $\partial_+\mathcal{L}^\theta(n)$ denote their nth. copies respectively. Taking $\mathcal{L}^\theta(0) = \mathcal{L}^\theta$, $\mathcal{T}^\theta(0) = \mathcal{T}^\theta$, $\partial_+\mathcal{L}^\theta(0) = \partial_+\mathcal{L}^\theta$, define inductively,

$$\begin{aligned} \tilde{\mathcal{X}}^\theta(0) &= \partial_- \mathcal{L}^\theta \\ \tilde{\mathcal{X}}^\theta(n+1) &= \tilde{\mathcal{X}}^\theta(n) \sqcup \mathcal{L}^\theta(n) \sqcup \partial_+\mathcal{L}^\theta(n) \sqcup \mathcal{T}^\theta(n) \sqcup \partial_-\mathcal{L}^\theta(n+1) \end{aligned}$$

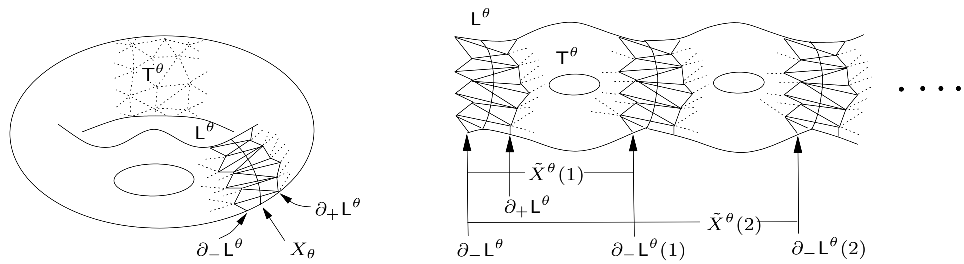

Figure 5: Complex X with level X_θ on left. Complexes $\tilde{X}^\theta(1)$, $\tilde{X}^\theta(2)$, $\cdots$ providing a filtration of the total space $\tilde{X}$ on right.

Let $I(\sigma,\tau)$ denote the incidence between σ and τ, that is, $I(\sigma,\tau) = 1$ if σ and τ are incident and 0 otherwise. Taking $I_0(\sigma,\tau) = I(\sigma,\tau)$, the incidences among the simplices are described by

$$\begin{aligned}
I_{n+1}(\sigma,\tau) &= I_n(\sigma,\tau) \text{ if } \sigma \in \tilde{\mathcal{X}}^\theta(n) \text{ and } \tau \text{ is a face of } \sigma \\
I_{n+1}(\sigma,\tau) &= I(\sigma,\tau) \text{ if } \sigma \in \mathcal{L}(n) \sqcup \partial_+\mathcal{L}^\theta(n) \subset \mathsf{L}^\theta \text{ and } \tau \text{ is a face of } \sigma \\
I_{n+1}(\sigma,\tau) &= I(\sigma,\tau) \text{ if } \sigma \in \mathcal{T}^\theta(n) \sqcup \partial_-\mathcal{L}^\theta(n+1) \subset \mathsf{T}^\theta \text{ and } \tau \text{ is a face of } \sigma.
\end{aligned}$$

In all other cases $I_{n+1}(\sigma,\tau) = 0$.

Notice that each $\tilde{\mathcal{X}}^\theta(i)$ forms a simplicial complex $\tilde{X}^\theta(i)$ (Figure 5). To describe $\tilde{f}$ it suffices to provide its values on vertices. We write

$$\mathcal{P} = \partial_-\mathcal{L}^\theta \sqcup \mathcal{L}^\theta \sqcup \partial_+\mathcal{L}^\theta \sqcup \mathcal{T}^\theta$$

and $\mathcal{P}_0$ for the subset of vertices in $\mathcal{P}$. We write $\mathcal{P}_0(n)$ for the $n-$th copy of $\mathcal{P}_0$ and define $\tilde{f}(n) : \mathcal{P}_0(n) \to \mathbb{R}$ by $\tilde{f}(n) := \tilde{f} + 2\pi n$ where $\tilde{f} = p^{-1} \cdot f$ with $p : (t-\pi, t+\pi] \to \mathbb{S}^1$ which [3] sends t to θ. Once defined on vertices, $\tilde{f}$ is extended by linearity to each simplex of the simplicial complex $\tilde{X}^\theta(n)$.

Note that an order of the simplices of X satisfying condition A induces an order on the simplices of $\mathcal{T}^\theta(n)$, $\partial_\pm\mathcal{L}^\theta(n)$ and $\mathcal{L}^\theta(n)$ and by juxtaposition an order on the simplices of $\tilde{X}^\theta(n)$ which continue to satisfy condition A. It implies that one can build a matrix $M(\tilde{X}^\theta(n))$ which satisfies condition A by juxtaposing the minors of $M(X)$ that represent $\mathcal{L}^\theta$, $\partial_\pm L^\theta$, $\mathcal{T}^\theta$, and their copies in an appropriate order. Note also that $\tilde{X}^\theta(n)$ is a sub complex of $\tilde{X}$ and therefore the restriction of $\tilde{f}$ provides tame maps on each of these spaces. The columns and rows of $M(\tilde{X}^\theta(n))$ can be reordered so that they become filtration compatible with $\tilde{f}$.

D Computing bar codes with standard persistence

In what follows we propose an alternative method to derive the bar codes using the standard persistence algorithm which computes the sub level persistence applied to various subspaces Y of X canonically derived from X and f as indicated in [1].

For this purpose we need also a minor extension of sublevel persistence, which we refer as simultaneous persistence. To describe it, we consider two maps $f^\pm : W^\pm \to [0,\infty)$ with the condition that $(f^\pm)^{-1}(0) = A \subset W^\pm$. Denote by $\omega^{f^-,f^+}(0;s,t)$ the maximal number of linearly independent elements in $H_r(A)$ which die with respect to f^- exactly at s and with respect to f^+ exactly at t. These numbers are analogues to the numbers $\mu_r(\cdots)$ considered for the sublevel persistence and can be computed by running the standard persistence algorithms for W^+ and W^-, see [1].

Let $f : X \to \mathbb{R}$ be a tame map with critical values $\cdots s_{i-1} < s_i < s_{i+1} < s_{i+2} \cdots$. We apply the discussion below to $\tilde{f} : \tilde{X} \to \mathbb{R}$.

[3] p is bijective and continuous, but p^{-1} is not continuous

For $s_i < s_j$, denote by

1. $N\{s_i, s_j) =$ the number of bar codes which intersect the level X_{s_i} with open end at s_j,
2. $N(s_i, s_j\} =$ the number of bar codes which intersect the level X_{s_j} with open end at s_i,
3. $N(s_i, s_j) =$ the number of bar codes which have both ends s_i and s_j open, and
4. $N\{s_i, s_j\} =$ the number of bar codes which intersect both levels X_{s_i} and X_{s_j}.

These numbers can be computed using the sublevel persistence and the simultaneous persistence as follows. For $s \in \mathbb{R}$ consider the maps $g_s^{\pm} : Y^{\pm}(s) \to [0, \infty)$ defined by:

$$
\begin{gathered}
Y^{+}(s) = X_{[s,\infty)}, Y^{-}(s) = X_{(-\infty,s]} \\
g_s^{+} = f|_{X_{[s,\infty)}} - s, g_s^{-} = -f|_{X_{(-\infty,s]}} + s.
\end{gathered}
$$

We have:

$$
\begin{aligned}
N\{s_i, s_j) =& \mu_r^{g_{s_i}^{+}}(0, s_j - s_i) \\
N(s_i, s_j\} =& \mu_r^{g_{s_j}^{-}}(0, s_j - s_i) \\
N(s_i, s_j) =& \omega^{g_s^{-}, g_s^{+}}(0, s - s_j, s - s_i),\ s \in (s_i, s_j).
\end{aligned}
$$

Hence, the cardinality of $[s_i, s_j)$ is

$$N[s_i, s_j) := N\{s_i, s_j) - N\{s_{i-1}, s_j) - N(s_i, s_j).$$

Similarly, the cardinality of $(s_i, s_j]$ is

$$N(s_i, s_j] = N(s_i, s_j\} - N(s_i, s_{j+1}\} - N(s_i, s_{j+1}).$$

and the cardinality of (s_i, s_j) equals $N(s_i, s_j)$. We are only left with computing the number of bar codes of type $[s_i, s_j]$.

Note that if we denote by

1. $N\{s_i, s_j] =$ the number of bar codes which intersect the level X_{s_i} and with closed end at s_j,
2. $N[s_i, s_j\} =$ the number of bar codes which intersect the levels X_{s_j} and with closed end at s_i,
3. $N[s_i, s_j] =$ the number of bar codes which have both ends s_i and s_j closed, and
4. $N\{s_i, s_j\} =$ the number of bar codes which intersect both levels X_{s_i} and X_{s_j},

then we have

$$N\{s_i, s_j] = N\{s_i, s_j\} - N\{s_i, s_{j+1}\} - N\{s_i, s_{j+1})$$

and

$$N[s_i, s_j\} = N\{s_i, s_j\} - N\{s_{i-1}, s_j\} - N(s_{i-1}, s_j\}$$

with the last terms $N\{s_i, s_{j+1})$ and $N(s_{i-1}, s_j\}$ being computed as discussed above and finally

$$N[s_i, s_j] = N\{s_i, s_j] - N\{s_{i-1}, s_j] - N(s_{i-1}, s_j].$$

E Decomposing a quiver representation

We present a procedure to calculate both the Jordan cells and the bar codes. The decomposition of each representation as a finite sum of indecomposables and the explicit description of each indecomposable as presented in section 4 imply the following. For each vector space V_i of the representation ρ (of G_{2m}), one can find a subspace V_i^J and the linearly independent vectors $\{e_i^1, e_i^2, \cdots e_i^{n_i}\}$ whose linear span V_i^B complements V_i^J (i.e. $V_i = V_i^B \oplus V_i^J$) so that α_i's and β_i's send V_{2i-1}^J and V_{2i+1}^J respectively to V_{2i}^J by isomorphisms, and send each $e_{2i-1}^{\cdots}$ and $e_{2i-1}^{\cdots}$ to either $e_{2i}^{\cdots}$ or to zero.

With this in mind we describe four types of transformations T_1^i, T_2^i, T_3^i, and T_4^i of the representation ρ into a representation ρ' as follows.

1. If $\rho' = T_1^i(\rho)$ then $V'_{2i+1} = V_{2i+1}/\ker(\beta_i)$, $V'_{2i+2} = V_{2i+2}/\alpha_{i+1}(\ker(\beta_i))$, $V'_r = V_r$ for $r \neq \{2i+1, 2i+2\}$ with α'_r, β'_r being induced from α_r, β_r.

2. If $\rho' = T_2^i(\rho)$ then $V'_{2i-1} = V_{2i-1}/\ker(\alpha_i)$, $V'_{2i-2} = V'_{2i-2}/\beta_{i-1}(\ker\alpha_i)$, $V'_r = V_r$ for $r \neq \{2i-1, 2i-2\}$ with α'_r, β'_r being induced from α_r, β_r.

   In both cases ρ' is a quotient representation of ρ.

3. If $\rho' = T_3^i(\rho)$ then $V'_{2i} = \alpha_i(V_{2i-1})$, $V'_{2i+1} = \beta_i^{-1}(V'_{2i})$, $V'_r = V_r$ for $r \neq \{2i, 2i+1\}$ with α'_r, β'_r being the restrictions of α_r, β_r.

4. If $\rho' = T_4^i(\rho)$ then $V'_{2i} = \beta_i(V_{2i+1})$, $V'_{2i-1} = \alpha_i^{-1}(V'_{2i})$ $V'_r = V_r$ for $r \neq \{2i, 2i-1\}$ with α'_r, β'_r being the restrictions of α_r, β_r.

   In these last two cases the representation ρ' is a sub representation of ρ.

Transformation T_1^i eliminates all bar codes of the form $(i, i+1)$ and $(i, i+1]$, transforms each bar code of the form $(i, k]$ and (i, k), $k \geq (i+2)$, into a bar code $(i+1, k]$ and $(i+1, k)$ respectively and leaves all other type of barcodes and Jordan cells unchanged.

Transformation T_2^i eliminates all bar codes of the form $(i-1, i)$ and $[i-1, i)$, transforms each bar code of the form $[l, i)$ and (l, i), $l \leq i-2$, into a bar code $[l, i-1)$ and $(l, i-1)$ respectively, and leaves any other type of barcodes and Jordan cells unchanged.

Transformation T_3^i eliminates all bar codes of the form $[i, i]$ and $[i, i+1)$,transforms each bar code of the form $[i, k)$, and $[i, k]$, $k \geq i+1$, into the bar code $[i+1, k)$ and $[i+1, k]$ respectively (assuming barcodes of the forms $[i+1, i+1)$ and $(i+1, i+1]$ do not exist), and leaves all other type of barcodes and Jordan cells unchanged.

Transformation T_4^i eliminates all bar codes of the form $[i, i]$ and $(i-1, i]$, transforms each bar code of the form $(l, i]$ and $[l, i]$, $l \leq i-1$, into the bar code $(l, i-1]$ and $[l, i-1]$ respectively, and leaves all other type of barcodes and Jordan cell unchanged.

After applying these transformations a finite number of times (at most $\sum \dim(V_i)$ times), one ends up with a representation with the same Jordan cells as the initial one and no bar codes, hence with all α's and β's being isomorphisms. Subsequently, one can compute all Jordan cells by applying Observation 4.1.

Notice that the knowledge of the bar codes of ρ' in each of the modifications described above and the multiplicity of the bar codes which have been eliminated determines entirely the bar codes of ρ.

The numbers of bar codes of type $(i, i+1)$ and $(i-1, i)$ eliminated by T_1^i and T_2^i respectively are

$$\dim(\alpha_{i+1}(\ker\beta_i) - \dim(\alpha_{i+1}(\ker\beta_i) \cap \beta_{i+1}(V_{2i+3}))$$

and

$$\dim(\beta_{i-1}(\ker\alpha_i) - \dim(\beta_{i-1}(\ker\alpha_i) \cap \alpha_{i-1}(V_{2i-3})) \text{ respectively.}$$

The number of bar codes of type $[i,i]$ eliminated by T_3^i and T_4^i is $\dim(V_{2i})-\dim(\alpha_i(V_{2i-1})+\beta_i(V_{2i+1}))$, while the numbers of bar codes of type $[i,1+1)$ and $(i-1,i]$ eliminated by T_3^i and T_4^i respectively are given by explicit formulae. However, if we apply the transformations T_1^i and T_2^i first, we get rid of all bar codes with at least one end open and the formulae mentioned above are not necessary.

F Jordan cells: an alternative method

We begin with the representation of G_{2m}

$$\rho = \{V_r, \alpha_i : V_{2i-1} \to V_{2i}, \beta_i : V_{2i+1} \to V_{2i} | 1 \le r \le 2m, 1 \le i \le m, V_{2m+1} = V_1\}$$

and define the sequences

$$\{V_r, r = 1, 2, \cdots \infty\}, \{\alpha_i : V_{2i-1} \to V_{2i}, i = 1, 2, \cdots \infty\}, \{\beta_i : V_{2i+1} \to V_{2i}, i = 1, 2 \cdots \infty\}$$

by

$$V_{r+2mk} := V_r, \quad \alpha_{i+mk} = \alpha_i, \quad \beta_{i+mk} = \beta_i, 1 \le r, i < \infty.$$

Inductively define

$$\{V'_r, \alpha'_i : V'_{2i-1} \to V'_{2i}, \beta'_i : V'_{2i+1} \to V'_{2i} | 1 \le r \le \infty, 1 \le i \le \infty\}$$

by

$$\begin{aligned} V'_1 =& V_1 \\ V'_{2i} =& \alpha_i(V'_{2i-1}) \\ V'_{2i+1} =& \beta_i^{-1}(V'_{2i}) \\ \alpha'_i =& \alpha_i|_{V'_{2i-1}} \\ \beta'_i =& \beta_i|_{V'_{2i+1}}. \end{aligned}$$

Observe that $V'_{r+2m(i+1)} \subseteq V'_{r+2mi}$ where equality holds for large enough i. Consequently, there is a k such that

$$\rho' := \{V'_{r+2mk}, \alpha'_{i+2mk} : V'_{(2i-1)+2mk} \to V'_{2i+2mk}, \beta'_{i+2mk} : V'_{(2i+1)+2mk} \to V'_{2i+2mk} | 1 \le r \le 2m, 1 \le i \le m\}$$

defines a representation of G_{2m} with α'_i, β'_i being isomorphisms. It is not hard to check that the Jordan cells of ρ and ρ' are the same. Observation 4.1 permits one to calculate the Jordan cells of ρ' hence of ρ.