

Topological Analysis of Nerves, Reeb Spaces, Mappers, and Multiscale Mappers

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Abstract

Data analysis often concerns not only the space where data come from, but also various types of maps attached to data. In recent years, several related structures have been used to study maps on data, including Reeb spaces, mappers and multiscale mappers. The construction of these structures also relies on the so-called *nerve* of a cover of the domain.

In this paper, we aim to analyze the topological information encoded in these structures in order to provide better understanding of these structures and facilitate their practical usage.

More specifically, we show that the one-dimensional homology of the nerve complex $N(\mathcal{U})$ of a path-connected cover \mathcal{U} of a domain X cannot be richer than that of the domain X itself. Intuitively, this result means that no new H_1 -homology class can be “created” under a natural map from X to the nerve complex $N(\mathcal{U})$. Equipping X with a pseudometric d , we further refine this result and characterize the classes of $H_1(X)$ that may survive in the nerve complex using the notion of *size* of the covering elements in \mathcal{U} . These fundamental results about nerve complexes then lead to an analysis of the H_1 -homology of Reeb spaces, mappers and multiscale mappers.

The analysis of H_1 -homology groups unfortunately does not extend to higher dimensions. Nevertheless, by using a map-induced metric, establishing a Gromov-Hausdorff convergence result between mappers and the domain, and interleaving relevant modules, we can still analyze the persistent homology groups of (multiscale) mappers to establish a connection to Reeb spaces.

1 Introduction

Data analysis often concerns not only the space where data come from, but also various types of information attached to data. For example, each node in a road network can contain information about the average traffic flow passing this point, a node in protein-protein interaction network can be associated with biochemical properties of the proteins involved. Such information attached to data can be modeled as maps defined on the domain

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of interest; note that the maps are not necessarily \mathbb{R}^d -valued, e.g, the co-domain can be \mathbb{S}^1 . Hence understanding data benefits from analyzing maps relating two spaces rather than a single space with no map on it.

In recent years, several related structures have been used to study general maps on data, including Reeb spaces [10, 12, 14, 19], mappers (and variants) [5, 9, 22] and multiscale mappers [11]. More specifically, given a map $f : X \rightarrow Z$ defined on a topological space X , the Reeb space R_f w.r.t. f (first studied for piecewise-linear maps in [14]), is a generalization of the so-called Reeb graph for a scalar function which has been used in various applications [2]. It is the quotient space of X w.r.t. an equivalence relation that asserts two points of X to be equivalent if they have the same function value and are connected to each other via points of the same function value. All equivalent points are collapsed into a single point in the Reeb space. Hence R_f provides a way to view X from the perspective of f .

The Mapper structure, originally introduced in [22], can be considered as a further generalization of the Reeb space. Given a map $f : X \rightarrow Z$, it also considers a cover \mathcal{U} of the co-domain Z that enables viewing the structure of f at a coarser level. Intuitively, the equivalence relation between points in X is now defined by whether points are within the same connected component of the pre-image of a cover element $U \in \mathcal{U}$. Instead of a quotient space, the mapper takes the nerve complex of the cover of X formed by the connected components of the pre-images of all elements in \mathcal{U} (i.e, the cover formed by those equivalent points). Hence the mapper structure provides a view of X from the perspective of both f and a cover of the co-domain Z .

Finally, both the Reeb space and the mapper structures provide a fixed snapshot of the input map f . As we vary the cover \mathcal{U} of the co-domain Z , we obtain a family of snapshots at different granularities. The *multiscale mapper* [11] describes the sequence of the mapper structures as one varies the granularity of the cover of Z through a sequence of covers of Z connected via cover maps.

New work. While these structures are meaningful in that they summarize the information contained in data, there has not been any qualitative analysis of the precise information encoded by them with the only exception of [5] and [15]¹. In this paper, we aim to analyze the *topological information* encoded by these structures, so as to provide better understanding of these structures and facilitate their practical usage [13, 18]. In particular, the construction of the mapper and multiscale mapper use the so-called *nerve* of a cover of the domain. To understand the mappers and multiscale mappers, we first provide a quantitative analysis of the topological information encoded in the nerve of a reasonably well-behaved cover for a domain. Given the generality and importance of the nerve complex in topological studies, this result is of independent interest.

More specifically, in Section 3, we first obtain a general result that relates the one dimensional homology H_1 of the nerve complex $N(\mathcal{U})$ of a path-connected cover \mathcal{U} (where each open set contained is path-connected) of a domain X to that of the domain X itself. Intuitively, this result says that no new H_1 -homology classes can be “created” under a natural

¹Carrière and Oudot [5] analyzed certain persistence diagram of mappers induced by a real-valued function, and provided a characterization for it in terms of the persistence diagram of the corresponding Reeb graph. Gasparovic et al [15] provides full description of the persistence homology information encoded in the *intrinsic Čech complex* (a special type of nerve complex) of a metric graph.

map from X to the nerve complex $N(\mathcal{U})$. Equipping X with a pseudometric d , we further refine this result and quantify the classes of $H_1(X)$ that may survive in the nerve complex (Theorem 21, Section 4). This demarcation is obtained via a notion of *size* of covering elements in \mathcal{U} . These fundamental results about nerve complexes then lead to an analysis of the H_1 -homology classes in Reeb spaces (Theorem 27), mappers and multiscale mappers (Theorem 29). The analysis of H_1 -homology groups unfortunately does not extend to higher dimensions. Nevertheless, we can still provide an interesting analysis of the persistent homology groups for these structures (Theorem 41, Section 5). During this course, by using a map-induced metric, we establish a Gromov-Hausdorff convergence between the mapper structure and the domain. This offers an alternative to [19] for defining the convergence between mappers and the Reeb space, which may be of independent interest.

2 Topological background and motivation

Space, paths, covers. Let X denote a path connected topological space. Since X is path connected, there exists a path $\gamma : [0, 1] \rightarrow X$ connecting every pair of points $\{x, x'\} \in X \times X$ where $\gamma(0) = x$ and $\gamma(1) = x'$. Let $\Gamma_X(x, x')$ denote the set of all such paths connecting x and x' . These paths play an important role in our definitions and arguments.

By a cover of X we mean a collection $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of open sets such that $\bigcup_{\alpha \in A} U_\alpha = X$. A cover \mathcal{U} is *path connected* if each U_α is path connected. In this paper, we consider only path connected covers.

Later to define maps between X and its nerve complexes, we need X to be *paracompact*, that is, every cover \mathcal{U} of X has a subcover $\mathcal{U}' \subseteq \mathcal{U}$ so that each point $x \in X$ has an open neighborhood contained in *finitely many* elements of \mathcal{U}' . Such a cover \mathcal{U}' is called *locally finite*. From now on, we assume X to be *compact* which implies that it is paracompact too.

Definition 1 (Simplicial complex and maps). *A simplicial complex K with a vertex set V is a collection of subsets of V with the condition that if $\sigma \in 2^V$ is in K , then all subsets of σ are in K . We denote the geometric realization of K by $|K|$. Let K and L be two simplicial complexes. A map $\phi : K \rightarrow L$ is simplicial if for every simplex $\sigma = \{v_1, v_2, \dots, v_p\}$ in K , the simplex $\phi(\sigma) = \{\phi(v_1), \phi(v_2), \dots, \phi(v_p)\}$ is in L .*

Definition 2 (Nerve of a cover). *Given a cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ of X , we define the nerve of the cover \mathcal{U} to be the simplicial complex $N(\mathcal{U})$ whose vertex set is the index set A , and where a subset $\{\alpha_0, \alpha_1, \dots, \alpha_k\} \subseteq A$ spans a k -simplex in $N(\mathcal{U})$ if and only if $U_{\alpha_0} \cap U_{\alpha_1} \cap \dots \cap U_{\alpha_k} \neq \emptyset$.*

Maps between covers. Given two covers $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{V_\beta\}_{\beta \in B}$ of X , a *map of covers* from \mathcal{U} to \mathcal{V} is a set map $\xi : A \rightarrow B$ so that $U_\alpha \subseteq V_{\xi(\alpha)}$ for all $\alpha \in A$. By a *slight abuse of notation* we also use ξ to indicate the map $\mathcal{U} \rightarrow \mathcal{V}$. Given such a map of covers, there is an induced simplicial map $N(\xi) : N(\mathcal{U}) \rightarrow N(\mathcal{V})$, given on vertices by the map ξ . Furthermore, if $\mathcal{U} \xrightarrow{\xi} \mathcal{V} \xrightarrow{\zeta} \mathcal{W}$ are three covers of X with the intervening maps of covers between them, then $N(\zeta \circ \xi) = N(\zeta) \circ N(\xi)$ as well. The following simple result is useful.

Proposition 3 (Maps of covers induce contiguous simplicial maps [11]). *Let $\zeta, \xi : \mathcal{U} \rightarrow \mathcal{V}$ be any two maps of covers. Then, the simplicial maps $N(\zeta)$ and $N(\xi)$ are contiguous.*

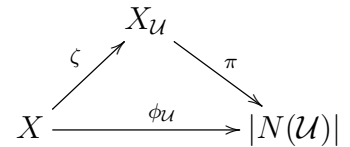
Recall that two simplicial maps $h_1, h_2 : K \rightarrow L$ are *contiguous* if for all $\sigma \in K$ it holds that $h_1(\sigma) \cup h_2(\sigma) \in L$. In particular, contiguous maps induce identical maps at the homology level [20]. Let $H_k(\cdot)$ denote the k -dimensional homology of the space in its argument. This homology is *singular* or *simplicial* depending on if the argument is a topological space or a simplicial complex respectively. All homology groups in this paper are defined over the field \mathbb{Z}_2 . Proposition 3 implies that the map $H_k(N(\mathcal{U})) \rightarrow H_k(N(\mathcal{V}))$ arising out of a cover map can be deemed canonical.

3 Surjectivity in H_1 -persistence

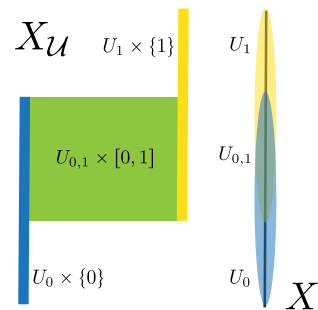
In this section we first establish a map $\phi_{\mathcal{U}}$ between X and the geometric realization $|N(\mathcal{U})|$ of a nerve complex $N(\mathcal{U})$. This helps us to define a map $\phi_{\mathcal{U}*}$ from the singular homology groups of X to the simplicial homology groups of $N(\mathcal{U})$ (through the singular homology of $|N(\mathcal{U})|$). The famous nerve theorem [4, 17] says that if the elements of \mathcal{U} intersect only in contractible spaces, then $\phi_{\mathcal{U}}$ is a homotopy equivalence and hence $\phi_{\mathcal{U}*}$ leads to an isomorphism between $H_*(X)$ and $H_*(N(\mathcal{U}))$. The contractibility condition can be weakened to a *homology ball* condition to retain the isomorphism between the two homology groups [17]. In absence of such conditions of the cover, simple examples exist to show that $\phi_{\mathcal{U}*}$ is neither a monomorphism (injection) nor an epimorphism (surjection). Figure 1 gives an example where $\phi_{\mathcal{U}*}$ is not surjective in H_2 . However, for one dimensional homology we show that, for any path connected cover \mathcal{U} , the map $\phi_{\mathcal{U}*}$ is necessarily a surjection. One implication of this is that the simplicial maps arising out of cover maps induce a surjection among the one dimensional homology groups of two nerve complexes.

3.1 Nerves

The proof of the nerve theorem [16] uses a construction that connects the two spaces X and $|N(\mathcal{U})|$ via a third space $X_{\mathcal{U}}$ that is a product space of \mathcal{U} and the geometric realization $|N(\mathcal{U})|$. In our case \mathcal{U} may not satisfy the contractibility condition. Nevertheless, we use the same construction to define three maps, $\zeta : X \rightarrow X_{\mathcal{U}}$, $\pi : X_{\mathcal{U}} \rightarrow |N(\mathcal{U})|$, and $\phi_{\mathcal{U}} : X \rightarrow |N(\mathcal{U})|$ where $\phi_{\mathcal{U}} = \pi \circ \zeta$ is referred to as the *nerve map*. Details about the construction of these maps follow.



Denote the elements of the cover \mathcal{U} as U_{α} for α taken from some indexing set A . The vertices of $N(\mathcal{U})$ are denoted by $\{u_{\alpha}, \alpha \in A\}$, where each u_{α} corresponds to the cover element U_{α} . For each finite non-empty intersection $U_{\alpha_0, \dots, \alpha_n} := \bigcap_{i=0}^n U_{\alpha_i}$ consider the product $U_{\alpha_0, \dots, \alpha_n} \times \Delta_{\alpha_0, \dots, \alpha_n}^n$, where $\Delta_{\alpha_0, \dots, \alpha_n}^n$ denotes the n -dimensional simplex with vertices $u_{\alpha_0}, \dots, u_{\alpha_n}$. Consider now the disjoint union



$$M := \bigsqcup_{\alpha_0, \dots, \alpha_n \in A: U_{\alpha_0, \dots, \alpha_n} \neq \emptyset} U_{\alpha_0, \dots, \alpha_n} \times \Delta_{\alpha_0, \dots, \alpha_n}^n$$

together with the following identification: each point $(x, y) \in M$, with $x \in U_{\alpha_0, \dots, \alpha_n}$ and $y \in [\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_n] \subset \Delta_{\alpha_0, \dots, \alpha_n}^n$ is identified with the corresponding

point in the product $U_{\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_n} \times \Delta_{\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_n}$ via the inclusion $U_{\alpha_0, \dots, \alpha_n} \subset U_{\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_n}$. Here $[\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_n]$ denotes the i -th face of the simplex $\Delta_{\alpha_0, \dots, \alpha_n}^n$. Denote by \sim this identification and now define the space $X_{\mathcal{U}} := M / \sim$. An example for the case when X is a line segment and \mathcal{U} consists of only two open sets is shown in the previous page.

Definition 4. A collection of real valued continuous functions $\{\varphi_\alpha : \rightarrow [0, 1], \alpha \in A\}$ is called a partition of unity if (i) $\sum_{\alpha \in A} \varphi_\alpha(x) = 1$ for all $x \in X$, (ii) For every $x \in X$, there are only finitely many $\alpha \in A$ such that $\varphi_\alpha(x) > 0$.

If $\mathcal{U} = \{U_\alpha, \alpha \in A\}$ is any open cover of X , then a partition of unity $\{\varphi_\alpha, \alpha \in A\}$ is subordinate to \mathcal{U} if $\text{supp}(\varphi_\alpha)$ is contained in U_α for each $\alpha \in A$.

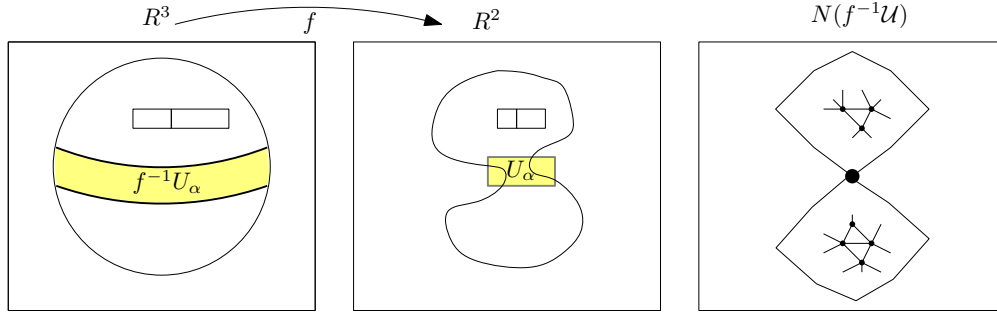


Figure 1: The map $f : \mathbb{S}^2 \subset \mathbb{R}^3 \rightarrow \mathbb{R}^2$ takes the sphere to \mathbb{R}^2 . The pullback of the cover element U_α makes a band surrounding the equator which causes the nerve $N(f^{-1}\mathcal{U})$ to pinch in the middle creating two 2-cycles. This shows that the map $\phi_* : X \rightarrow N(*)$ may not induce a surjection in H_2 .

Since X is paracompact, for any open cover $\mathcal{U} = \{U_\alpha, \alpha \in A\}$ of X , there exists a partition of unity $\{\varphi_\alpha, \alpha \in A\}$ subordinate to \mathcal{U} [21]. For each $x \in X$ such that $x \in U_\alpha$, denote by x_α the corresponding copy of x residing in $X_{\mathcal{U}}$. Then, the map $\zeta : X \rightarrow X_{\mathcal{U}}$ is defined as follows: for any $x \in X$,

$$\zeta(x) := \sum_{\alpha \in A} \varphi_\alpha(x) x_\alpha.$$

The map $\pi : X_{\mathcal{U}} \rightarrow |N(\mathcal{U})|$ is induced by the individual projection maps

$$U_{\alpha_0, \dots, \alpha_n} \times \Delta_{\alpha_0, \dots, \alpha_n}^n \rightarrow \Delta_{\alpha_0, \dots, \alpha_n}^n.$$

Then, it follows that $\phi_{\mathcal{U}} = \pi \circ \zeta : X \rightarrow |N(\mathcal{U})|$ satisfies, for $x \in X$,

$$\phi_{\mathcal{U}}(x) = \sum_{\alpha \in A} \varphi_\alpha(x) u_\alpha. \quad (1)$$

We have the following fact [21, pp. 108]:

Fact 5. ζ is a homotopy equivalence.

3.2 From space to nerves

Now, we show that the nerve maps at the homology level are surjective for one dimensional homology when the covers are path-connected. Interestingly, the result is not true beyond one dimensional homology (see Figure 1) which is probably why this simple but important fact has not been observed before. First, we make a simple observation that connects the classes in singular homology of $|N(\mathcal{U})|$ to those in the simplicial homology of $N(\mathcal{U})$. The result follows immediately from the isomorphism between singular and simplicial homology induced by the geometric realization; see [20, Theorem 34.3]. In what follows let $[c]$ denote the class of a cycle c .

Proposition 6. *Every 1-cycle ξ in $|N(\mathcal{U})|$ has a 1-cycle γ in $N(\mathcal{U})$ so that $[\xi] = [[\gamma]]$.*

Proposition 7. *If \mathcal{U} is path connected, $\phi_{\mathcal{U}*} : H_1(X) \rightarrow H_1(|N(\mathcal{U})|)$ is a surjection.*

Proof. Let $[\gamma]$ be any class in $H_1(|N(\mathcal{U})|)$. Because of Proposition 6, we can assume that $\gamma = |\gamma'|$, where γ' is a 1-cycle in the 1-skeleton of $N(\mathcal{U})$. We construct a 1-cycle $\gamma_{\mathcal{U}}$ in $X_{\mathcal{U}}$ so that $\pi(\gamma_{\mathcal{U}}) = \gamma$. Recall the map $\zeta : X \rightarrow X_{\mathcal{U}}$ in the construction of the nerve map $\phi_{\mathcal{U}}$ where $\phi_{\mathcal{U}} = \pi \circ \zeta$. There exists a class $[\gamma_X]$ in $H_1(X)$ so that $\zeta_*([\gamma_X]) = [\gamma_{\mathcal{U}}]$ because ζ_* is an isomorphism by Fact 5. Then, $\phi_{\mathcal{U}*}([\gamma_X]) = \pi_*(\zeta_*([\gamma_X]))$ because $\phi_{\mathcal{U}*} = \pi_* \circ \zeta_*$. It follows $\phi_{\mathcal{U}*}([\gamma_X]) = \pi_*([\gamma_{\mathcal{U}}]) = [\gamma]$ showing that $\phi_{\mathcal{U}*}$ is surjective.

Therefore, it remains only to show that a 1-cycle $\gamma_{\mathcal{U}}$ can be constructed given γ' in $N(\mathcal{U})$ so that $\pi(\gamma_{\mathcal{U}}) = \gamma = |\gamma'|$. Let $e_0, e_1, \dots, e_{r-1}, e_r = e_0$ be an ordered sequence of edges on γ . Recall the construction of the space $X_{\mathcal{U}}$. In that terminology, let $e_i = \Delta_{\alpha_i \alpha_{(i+1) \bmod r}}^n$. Let $v_i = e_{(i-1) \bmod r} \cap e_i$ for $i \in [0, r-1]$. The vertex $v_i = v_{\alpha_i}$ corresponds to the cover element U_{α_i} where $U_{\alpha_i} \cap U_{\alpha_{(i+1) \bmod r}} \neq \emptyset$ for every $i \in [0, r-1]$. Choose a point x_i in the common intersection $U_{\alpha_i} \cap U_{\alpha_{(i+1) \bmod r}}$ for every $i \in [0, r-1]$. Then, the *edge* path $\tilde{e}_i = e_i \times x_i$ is in $X_{\mathcal{U}}$ by construction. Also, letting x_{α_i} to be the lift of x_i in the lifted U_{α_i} , we can choose a *vertex* path $x_{\alpha_i} \rightsquigarrow x_{\alpha_{(i+1) \bmod r}}$ residing in the lifted U_{α_i} and hence in $X_{\mathcal{U}}$ because U_{α_i} is path connected. Consider the following cycle obtained by concatenating the edge and vertex paths

$$\gamma_{\mathcal{U}} = \tilde{e}_0 x_{\alpha_0} \rightsquigarrow x_{\alpha_1} \tilde{e}_1 \cdots \tilde{e}_{r-1} x_{\alpha_{r-1}} \rightsquigarrow x_{\alpha_0}$$

By projection, we have $\pi(\tilde{e}_i) = e_i$ for every $i \in [0, r-1]$ and $\pi(x_{\alpha_i} \rightsquigarrow x_{\alpha_{(i+1) \bmod r}}) = v_{\alpha_i}$ and thus $\pi(\gamma_{\mathcal{U}}) = \gamma$ as required. \square

Since we are eventually interested in the simplicial homology groups of the nerves rather than the singular homology groups of their geometric realizations, we make one more transition using the known isomorphism between the two homology groups. Specifically, if $\iota_{\mathcal{U}} : H_k(|N(\mathcal{U})|) \rightarrow H_k(N(\mathcal{U}))$ denotes this isomorphism, we let $\bar{\phi}_{\mathcal{U}*}$ denote the composition $\iota_{\mathcal{U}} \circ \phi_{\mathcal{U}*}$. As a corollary to Proposition 7, we obtain:

Theorem 8. *If \mathcal{U} is path connected, $\bar{\phi}_{\mathcal{U}*} : H_1(X) \rightarrow H_1(N(\mathcal{U}))$ is a surjection.*

3.3 From nerves to nerves

In this section we extend the result in Theorem 8 to simplicial maps between two nerves induced by cover maps. The following proposition is key to establishing the result.

Proposition 9 (Coherent partitions of unity). *Suppose $\{U_\alpha\}_{\alpha \in A} = \mathcal{U} \xrightarrow{\theta} \mathcal{V} = \{V_\beta\}_{\beta \in B}$ are open covers of the paracompact topological space X and $\theta : A \rightarrow B$ is a map of covers. Then there exists a partition of unity $\{\varphi_\alpha\}_{\alpha \in A}$ subordinate to the cover \mathcal{U} such that if for each $\beta \in B$ we define*

$$\psi_\beta := \begin{cases} \sum_{\alpha \in \theta^{-1}(\beta)} \varphi_\alpha & \text{if } \beta \in \text{im}(\theta); \\ 0 & \text{otherwise.} \end{cases}$$

then the set of functions $\{\psi_\beta\}_{\beta \in B}$ is a partition of unity subordinate to the cover \mathcal{V} .

Proof. The proof closely follows that of [21, Corollary pp. 97]. Since X is paracompact, there exists a locally finite refinement $\mathcal{W} = \{W_\lambda\}_{\lambda \in L}$ of \mathcal{U} , a refinement map $L \xrightarrow{\xi} A$, and a partition of unity $\{\omega_\lambda\}_{\lambda \in L}$ subordinate to \mathcal{W} . For each $\alpha \in A$ define

$$\varphi_\alpha := \begin{cases} \sum_{\lambda \in \xi^{-1}(\alpha)} \omega_\lambda & \text{if } \alpha \in \text{im}(\xi); \\ 0 & \text{otherwise.} \end{cases}$$

The fact that the sum is well defined and continuous follows from the fact that \mathcal{W} is locally finite. Let $C_\alpha := \bigcup_{\lambda \in \xi^{-1}(\alpha)} \text{supp}(\omega_\lambda)$. The set C_α is closed, $C_\alpha \subset U_\alpha$, and $\varphi_\alpha(x) = 0$ for $x \notin C_\alpha$ so that $\text{supp}(\varphi_\alpha) \subset C_\alpha \subset U_\alpha$. Now, to check that the family $\{C_\alpha\}_{\alpha \in A}$ is locally finite pick any point $x \in X$. Since \mathcal{W} is locally finite there is an open set O containing x such that O intersects only finitely many elements in \mathcal{W} . Denote these cover elements by $W_{\lambda_1}, \dots, W_{\lambda_N}$. Now, notice if $\alpha \in A$ and $\alpha \notin \{\xi(\lambda_i), i = 1, \dots, N\}$, then O does not intersect C_α . Then, the family $\{\text{supp}(\varphi_\alpha)\}_{\alpha \in A}$ is locally finite. It then follows that for $x \in X$ one has

$$\sum_{\alpha \in A} \varphi_\alpha(x) = \sum_{\alpha \in A} \sum_{\lambda \in \xi^{-1}(\alpha)} \omega_\lambda(x) = \sum_{\lambda \in L} \omega_\lambda(x) = 1.$$

We have obtained that $\{\varphi_\alpha\}_{\alpha \in A}$ is a partition of unity subordinate to \mathcal{U} . Now, the same argument can be applied to the family $\{\psi_\beta\}_{\beta \in B}$ to obtain the proof of the proposition. \square

Let $\{U_\alpha\}_{\alpha \in A} = \mathcal{U} \xrightarrow{\theta} \mathcal{V} = \{V_\beta\}_{\beta \in B}$ be two open covers of X connected by a map of covers. Apply Proposition 9 to obtain coherent partitions of unity $\{\varphi_\alpha\}_{\alpha \in A}$ and $\{\psi_\beta\}_{\beta \in B}$ subordinate to \mathcal{U} and \mathcal{V} , respectively. Let the nerve maps $\phi_{\mathcal{U}} : X \rightarrow |N(\mathcal{U})|$ and $\phi_{\mathcal{V}} : X \rightarrow |N(\mathcal{V})|$ be defined as in (1) above. Let $N(\mathcal{U}) \xrightarrow{\tau} N(\mathcal{V})$ be the simplicial map induced by the cover map θ . Then, τ can be extended to a continuous map $\hat{\tau}$ on the image of $\phi_{\mathcal{U}}$ as follows: for $x \in X$, $\hat{\tau}(\phi_{\mathcal{U}}(x)) = \sum_{\alpha \in A} \varphi_\alpha(x) v_{\theta(\alpha)}$.

Proposition 10. *Let \mathcal{U} and \mathcal{V} be two covers of X connected by a cover map $\mathcal{U} \xrightarrow{\theta} \mathcal{V}$. Then, the nerve maps $\phi_{\mathcal{U}}$ and $\phi_{\mathcal{V}}$ satisfy $\phi_{\mathcal{V}} = \hat{\tau} \circ \phi_{\mathcal{U}}$ where $\tau : N(\mathcal{U}) \rightarrow N(\mathcal{V})$ is the simplicial map induced by the cover map θ .*

Proof. For any point $p \in \text{im}(\phi_{\mathcal{U}})$, there is $x \in X$ where $p = \phi_{\mathcal{U}}(x) = \sum_{\alpha \in A} \varphi_\alpha(x) u_\alpha$. Then,

$$\begin{aligned} \hat{\tau} \circ \phi_{\mathcal{U}}(x) &= \hat{\tau} \left(\sum_{\alpha \in A} \varphi_\alpha(x) u_\alpha \right) = \sum_{\alpha \in A} \varphi_\alpha(x) \tau(u_\alpha) = \sum_{\alpha \in A} \varphi_\alpha(x) v_{\theta(\alpha)} \\ &= \sum_{\beta \in B} \sum_{\alpha \in \theta^{-1}(\beta)} \varphi_\alpha(x) v_{\theta(\alpha)} = \sum_{\beta \in B} \psi_\beta(x) v_\beta = \phi_{\mathcal{V}}(x) \end{aligned}$$

\square

An immediate corollary of the above Proposition is:

Corollary 11. *The induced maps of $\phi_{\mathcal{U}*} : H_k(X) \rightarrow H_k(|N(\mathcal{U})|)$, $\phi_{\mathcal{V}*} : H_k(X) \rightarrow H_k(|N(\mathcal{V})|)$, and $\hat{\tau}_* : H_k(|N(\mathcal{U})|) \rightarrow H_k(|N(\mathcal{V})|)$ at the homology levels commute, that is, $\phi_{\mathcal{V}*} = \hat{\tau}_* \circ \phi_{\mathcal{U}*}$.*

With transition from singular to simplicial homology, Corollary 11 implies that:

Proposition 12. *$\bar{\phi}_{\mathcal{V}*} = \tau_* \circ \bar{\phi}_{\mathcal{U}*}$ where $\bar{\phi}_{\mathcal{V}*} : H_k(X) \rightarrow H_k(N(\mathcal{V}))$, $\bar{\phi}_{\mathcal{U}*} : H_k(X) \rightarrow H_k(N(\mathcal{U}))$ and $\tau : N(\mathcal{U}) \rightarrow N(\mathcal{V})$ is the simplicial map induced by a cover map $\mathcal{U} \rightarrow \mathcal{V}$.*

Proposition 12 extends Theorem 8 to the simplicial maps between two nerves.

Theorem 13. *Let $\tau : N(\mathcal{U}) \rightarrow N(\mathcal{V})$ be a simplicial map induced by a cover map $\mathcal{U} \rightarrow \mathcal{V}$ where both \mathcal{U} and \mathcal{V} are path connected. Then, $\tau_* : H_1(N(\mathcal{U})) \rightarrow H_1(N(\mathcal{V}))$ is a surjection.*

Proof. Consider the maps

$$H_1(X) \xrightarrow{\bar{\phi}_{\mathcal{U}*}} H_1(N(\mathcal{U})) \xrightarrow{\tau_*} H_1(N(\mathcal{V})), \text{ and } H_1(X) \xrightarrow{\bar{\phi}_{\mathcal{V}*}} H_1(N(\mathcal{V})).$$

By Proposition 12, $\tau_* \circ \bar{\phi}_{\mathcal{U}*} = \bar{\phi}_{\mathcal{V}*}$. By Theorem 8, the map $\bar{\phi}_{\mathcal{V}*}$ is a surjection. It follows that τ_* is a surjection. \square

3.4 Mapper and multiscale mapper

In this section we extend the previous results to the structures called mapper and multiscale mapper. Recall that X is assumed to be compact. Consider a cover of X obtained indirectly as a pullback of a cover of another space Z . This gives rise to the so called *Mapper* and *Multiscale Mapper*. Let $f : X \rightarrow Z$ be a continuous map where Z is equipped with an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ for some index set A . Since f is continuous, the sets $\{f^{-1}(U_\alpha), \alpha \in A\}$ form an open cover of X . For each α , we can now consider the decomposition of $f^{-1}(U_\alpha)$ into its path connected components, so we write $f^{-1}(U_\alpha) = \bigcup_{i=1}^{j_\alpha} V_{\alpha,i}$, where j_α is the number of path connected components $V_{\alpha,i}$'s in $f^{-1}(U_\alpha)$. We write $f^*\mathcal{U}$ for the cover of X obtained this way from the cover \mathcal{U} of Z and refer to it as the *pullback* cover of X induced by \mathcal{U} via f . Note that by its construction, this pullback cover $f^*\mathcal{U}$ is path-connected.

Notice that there are pathological examples of f where $f^{-1}(U_\alpha)$ may shatter into infinitely many path components. This motivates us to consider *well-behaved* functions f : we require that for every path connected open set $U \subseteq Z$, the preimage $f^{-1}(U)$ has *finitely* many open path connected components. Henceforth, all such functions are assumed to be well-behaved.

Definition 14 (Mapper [22]). *Let $f : X \rightarrow Z$ be a continuous map. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of Z . The mapper arising from these data is defined to be the nerve simplicial complex of the pullback cover: $M(\mathcal{U}, f) := N(f^*\mathcal{U})$.*

When we consider a continuous map $f : X \rightarrow Z$ and we are given a map of covers $\xi : \mathcal{U} \rightarrow \mathcal{V}$ between covers of Z , we observed in [11] that there is a corresponding map of covers between the respective pullback covers of X : $f^*(\xi) : f^*\mathcal{U} \rightarrow f^*\mathcal{V}$. Furthermore, if $\mathcal{U} \xrightarrow{\xi} \mathcal{V} \xrightarrow{\theta} \mathcal{W}$ are three different covers of a topological space with the intervening maps of covers between them, then $f^*(\theta \circ \xi) = f^*(\theta) \circ f^*(\xi)$.

In the definition below, *objects* can be covers, simplicial complexes, or vector spaces.

Definition 15 (Tower). A tower \mathfrak{W} with resolution $r \in \mathbb{R}$ is any collection $\mathfrak{W} = \{\mathcal{W}_\varepsilon\}_{\varepsilon \geq r}$ of objects \mathcal{W}_ε indexed in \mathbb{R} together with maps $w_{\varepsilon, \varepsilon'} : \mathcal{W}_\varepsilon \rightarrow \mathcal{W}_{\varepsilon'}$ so that $w_{\varepsilon, \varepsilon} = \text{id}$ and $w_{\varepsilon', \varepsilon''} \circ w_{\varepsilon, \varepsilon'} = w_{\varepsilon, \varepsilon''}$ for all $r \leq \varepsilon \leq \varepsilon' \leq \varepsilon''$. Sometimes we write $\mathfrak{W} = \{\mathcal{W}_\varepsilon \xrightarrow{w_{\varepsilon, \varepsilon'}} \mathcal{W}_{\varepsilon'}\}_{r \leq \varepsilon \leq \varepsilon'}$ to denote the collection with the maps. Given such a tower \mathfrak{W} , $\text{res}(\mathfrak{W})$ refers to its resolution.

When \mathfrak{W} is a collection of covers equipped with maps of covers between them, we call it a tower of covers. When \mathfrak{W} is a collection of simplicial complexes equipped with simplicial maps between them, we call it a tower of simplicial complexes.

The pullback properties described at the end of section 2 make it possible to take the pullback of a given tower of covers of a space via a given continuous function into another space, so that we obtain the following.

Proposition 16 ([11]). Let $\mathfrak{U} = \{\mathcal{U}_\varepsilon\}$ be a tower of covers of Z and $f : X \rightarrow Z$ be a continuous function. Then, $f^*\mathfrak{U} = \{f^*\mathcal{U}_\varepsilon\}$ is a tower of (path-connected) covers of X .

In general, given a tower of covers \mathfrak{W} of a space X , the nerve of each cover in \mathfrak{W} together with each map of \mathfrak{W} provides a tower of simplicial complexes which we denote by $N(\mathfrak{W})$.

Definition 17 (Multiscale Mapper [11]). Let $f : X \rightarrow Z$ be a continuous map. Let \mathfrak{U} be a tower of covers of Z . Then, the multiscale mapper is defined to be the tower of the nerve simplicial complexes of the pullback: $\text{MM}(\mathfrak{U}, f) := N(f^*\mathfrak{U})$.

As we indicated earlier, in general, no surjection between X and its nerve may exist at the homology level. It follows that the same is true for the mapper $N(f^*\mathfrak{U})$. But for H_1 , we can apply the results contained in previous section to claim the following.

Theorem 18. Consider the following multiscale mapper arising out of a tower of path connected covers:

$$N(f^*\mathcal{U}_0) \rightarrow N(f^*\mathcal{U}_1) \rightarrow \cdots \rightarrow N(f^*\mathcal{U}_n)$$

- There is a surjection from $H_1(X)$ to $H_1(N(f^*\mathcal{U}_i))$ for each $i \in [0, n]$.
- Consider a H_1 -persistence module of a multiscale mapper as shown below.

$$H_1(N(f^*\mathcal{U}_0)) \rightarrow H_1(N(f^*\mathcal{U}_1)) \rightarrow \cdots \rightarrow H_1(N(f^*\mathcal{U}_n)) \quad (2)$$

All connecting maps in the above module are surjections.

The above result implies that, as we proceed forward through the multiscale mapper, no new homology classes are born. They can only die. Consequently, all bar codes in the persistence diagram of the H_1 -persistence module induced by it have the left endpoint at 0.

4 Analysis of persistent H_1 -classes

Using the language of persistent homology, the results in the previous section imply that one dimensional homology classes can die in the nerves, but they cannot be born. In this section, we analyze further to identify the classes that survive. The distinction among the classes is made via a notion of ‘size’. Intuitively, we show that the classes with ‘size’ much larger than the ‘size’ of the cover survive. The ‘size’ is defined with the pseudometric that the space X is assumed to be equipped with. Precise statements are made in the subsections.

4.1 H_1 -classes of nerves of pseudometric spaces

Let (X, d) be a pseudometric space, that is, d satisfies the axioms of a metric except that $d(x, x') = 0$ may not necessarily imply $x = x'$. Assume X to be compact as before. We define a ‘size’ for a homology class that reflects how big the smallest generator in the class is in the metric d .

Definition 19. *The size $s(X')$ of a subset X' of the pseudometric space (X, d) is defined to be its diameter, that is, $s(X') = \sup_{x, x' \in X'} d(x, x')$. The size of a class $c \in H_k(X)$ is defined as $s(c) = \inf_{z \in c} s(z)$.*

Definition 20. *A set of k -cycles z_1, z_2, \dots, z_n of $H_k(X)$ is called a generator basis if the classes $[z_1], [z_2], \dots, [z_n]$ together form a basis of $H_k(X)$. It is called a minimal generator basis if $\sum_{i=1}^n s(z_i)$ is minimal among all generator bases.*

Lebesgue number of a cover. Our goal is to characterize the classes in the nerve of \mathcal{U} with respect to the sizes of their preimages in X via the map $\phi_{\mathcal{U}}$ where \mathcal{U} is assumed to be path connected. The Lebesgue number of such a cover \mathcal{U} becomes useful in this characterization. It is the largest number $\lambda(\mathcal{U})$ so that any subset of X with size at most $\lambda(\mathcal{U})$ is contained in at least one element of \mathcal{U} . Formally,

$$\lambda(\mathcal{U}) = \sup\{\delta \mid \forall X' \subseteq X \text{ with } s(X') \leq \delta, \exists U_{\alpha} \in \mathcal{U} \text{ where } U_{\alpha} \supseteq X'\}$$

In the above definition, we can assume X' to be path-connected because if it were not, then a connected superset containing all components of X' is contained in U_{α} because U_{α} is path connected itself. We observe that a homology class of size no more than $\lambda(\mathcal{U})$ cannot survive in the nerve. Further, the homology classes whose sizes are significantly larger than the maximum size of a cover do necessarily survive where we define the maximum size of a cover as $s_{max}(\mathcal{U}) := \max_{U \in \mathcal{U}} \{s(U)\}$.

Let z_1, z_2, \dots, z_g be a non-decreasing sequence of the generators with respect to their sizes in a minimal generator basis of $H_1(X)$. Consider the map $\phi_{\mathcal{U}} : X \rightarrow |N(\mathcal{U})|$ as introduced in Section 3. We have the following result.

Theorem 21. *Let \mathcal{U} be a path-connected cover of X .*

- i. Let $\ell = g + 1$ if $\lambda(\mathcal{U}) > s(z_g)$. Otherwise, let $\ell \in [1, g]$ be the smallest integer so that $s(z_{\ell}) > \lambda(\mathcal{U})$. If $\ell \neq 1$, the class $\bar{\phi}_{\mathcal{U}*}[z_j] = 0$ for $j = 1, \dots, \ell - 1$. Moreover, if $\ell \neq g + 1$, the classes $\{\bar{\phi}_{\mathcal{U}*}[z_j]\}_{j=\ell, \dots, g}$ generate $H_1(N(\mathcal{U}))$.*
- ii. The classes $\{\bar{\phi}_{\mathcal{U}*}[z_j]\}_{j=\ell, \dots, g}$ are linearly independent where $s(z_{\ell}) > 4s_{max}(\mathcal{U})$.*

The result above says that only the classes of $H_1(X)$ generated by generators of large enough size survive in the nerve. To prove this result, we use a map ρ that sends each 1-cycle in $N(\mathcal{U})$ to a 1-cycle in X . We define a chain map $\rho : \mathcal{C}_1(N(\mathcal{U})) \rightarrow \mathcal{C}_1(X)$ among one dimensional chain groups as follows ². It is sufficient to exhibit the map for an elementary

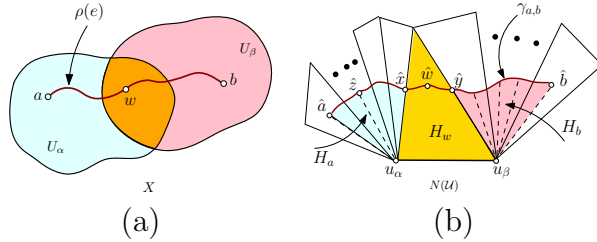
²We note that the high level framework of defining such a chain map and analyzing what it does to homologous cycles is similar to the work by Gasparovic et al. [15]. The technical details are different.

chain of an edge, say $e = \{u_\alpha, u_{\alpha'}\} \in \mathcal{C}_1(N(\mathcal{U}))$. Since e is an edge in $N(\mathcal{U})$, the two cover elements U_α and $U_{\alpha'}$ in X have a common intersection. Let $a \in U_\alpha$ and $b \in U_{\alpha'}$ be two points that are arbitrary but fixed for U_α and $U_{\alpha'}$ respectively. Pick a path $\xi(a, b)$ (viewed as a singular chain) in the union of U_α and $U_{\alpha'}$ which is path connected as both U_α and $U_{\alpha'}$ are. Then, define $\rho(e) = \xi(a, b)$. The following properties of $\phi_{\mathcal{U}}$ and ρ turn out to be useful.

Proposition 22. *Let γ be any 1-cycle in $N(\mathcal{U})$. Then, $[\phi_{\mathcal{U}}(\rho(\gamma))] = [|\gamma|]$.*

Proof. Let $e = (u_\alpha, u_\beta)$ be an edge in γ with u_α and u_β corresponding to U_α and U_β respectively. Let a and b be the corresponding fixed points for set U_α and U_β respectively. Consider the path $\rho(e) = \xi(a, b)$ in X as constructed above, and set $\gamma_{a,b} = \phi_{\mathcal{U}}(\xi(a, b))$ to be the image of $\rho(e)$ in $|N(\mathcal{U})|$. See Figure 2 for an illustration. Given an oriented path ℓ and two points $x, y \in \ell$, we use $\ell[x, y]$ to denote the subpath of ℓ from x to y . For a point $x \in X$, for simplicity we set $\hat{x} = \phi_{\mathcal{U}}(x)$ to be its image in $|N(\mathcal{U})|$.

Now, let $w \in \rho(e)$ be a point in $U_\alpha \cap U_\beta$, and $\hat{w} = \phi_{\mathcal{U}}(w)$ be its image in $\gamma_{a,b}$. We have the following observations. First, any point from $\gamma_{a,b}[\hat{a}, \hat{w}]$ is contained in a simplex in $N(\mathcal{U})$ incident on u_α . Similarly, any point from $\gamma_{a,b}[\hat{w}, \hat{b}]$ is contained in a simplex in $N(\mathcal{U})$ incident on u_β . These claims simply follow from the facts that $\rho(e)[a, w] \subset U_\alpha$ and $\rho(e)[w, b] \subset U_\beta$. Furthermore, let $\sigma_w \in N(\mathcal{U})$ be the lowest-dimensional simplex containing \hat{w} . Depending on the partition of unity that induces the map $\phi_{\mathcal{U}} : X \rightarrow |N(\mathcal{U})|$, it is possible that u_α and u_β are **not** vertices of σ_w .



However, as w is contained in each of the cover element from \mathcal{U} corresponding to the vertices of σ_w , and $w \in U_\alpha \cap U_\beta$, it must be contained in the common intersection of all these cover elements; thus there must exist simplex $\bar{\sigma}_w \in N(\mathcal{U})$ spanned by $Vert(\sigma_w) \cup \{u_\alpha, u_\beta\}$.

To this end, let $\gamma_{a,b}[\hat{x}, \hat{y}]$ be the maximal subpath of $\gamma_{a,b}$ containing \hat{w} that is contained within $|\bar{\sigma}_w|$. We assume that $\hat{x} \neq \hat{y}$ – The case where $\hat{x} = \hat{y}$ can be handled by a perturbation argument which we omit here.

Since the path $\gamma_{a,b}[\hat{a}, \hat{x}]$ is contained within a union of simplices all incident to the vertex u_α , one can construct a homotopy H_a that takes $\gamma_{a,b}[\hat{a}, \hat{x}]$ to u_α under which any point $\hat{z} \in \gamma_{a,b}[\hat{a}, \hat{x}]$ moves monotonically along the segment $\hat{z}u_\alpha$ within the geometric realization of the simplex containing both \hat{z} and u_α . See Figure 2 (b) where we draw a simple case for illustration. Similarly, there is a homotopy H_b that takes $\gamma_{a,b}[\hat{y}, \hat{b}]$ to u_β under which any point $\hat{z} \in \gamma_{a,b}[\hat{y}, \hat{b}]$ moves monotonically along the segment $\hat{z}u_\beta$. Finally, for the middle subpath $\gamma_{a,b}[\hat{x}, \hat{y}]$, since it is within simplex $|\bar{\sigma}_w|$ with $e = (u_\alpha, u_\beta)$ being an edge of it, we can construct homotopy H_w that takes $\gamma_{a,b}[\hat{x}, \hat{y}]$ to $u_\alpha u_\beta$ under which \hat{x} and \hat{y} move monotonically along the segments $\hat{x}u_\alpha$ and $\hat{y}u_\beta$ within the geometric realization of simplex $\bar{\sigma}_w$, respectively. Concatenating H_a , H_w and H_b , we obtain a homotopy $H_{\alpha,\beta}$ taking $\gamma_{a,b}$ to $|e|$. Therefore, a concatenation of these homotopies $H_{\alpha,\beta}$ considered over all edges in γ , brings $\phi_{\mathcal{U}}(\rho(\gamma))$ to $|\gamma|$ with a homotopy in $|N(\mathcal{U})|$. Hence, their homology classes are the same. \square

Proposition 23. *Let z be a 1-cycle in $\mathcal{C}_1(X)$. Then, $[\phi_{\mathcal{U}}(z)] = 0$ if $\lambda(\mathcal{U}) > s(z)$.*

Proof. It follows from the definition of the Lebesgue number that there exists a cover element $U_\alpha \in \mathcal{U}$ so that $z \subseteq U_\alpha$ because $s(z) < \lambda(\mathcal{U})$. We claim that there is a homotopy equivalence that sends $\phi_{\mathcal{U}}(z)$ to a vertex in $N(\mathcal{U})$ and hence $[\phi_{\mathcal{U}}(z)]$ is trivial.

Let x be any point in z . Recall that $\phi_{\mathcal{U}}(x) = \sum_i \varphi_i(x) u_{\alpha_i}$. Since U_α has a common intersection with each U_{α_i} so that $\varphi_{\alpha_i}(x) \neq 0$, we can conclude that $\phi_{\mathcal{U}}(x)$ is contained in a simplex with the vertex u_α . Continuing this argument with all points of z , we observe that $\phi_{\mathcal{U}}(z)$ is contained in simplices that share the vertex u_α . It follows that there is a homotopy that sends $\phi_{\mathcal{U}}(z)$ to u_α , a vertex of $N(\mathcal{U})$. \square

Proof of Theorem 21. Proof of (i): By Proposition 23, we have $\phi_{\mathcal{U}*}[z] = [\phi_{\mathcal{U}}(z)] = 0$ if $\lambda(\mathcal{U}) > s(z)$. This establishes the first part of the assertion because $\bar{\phi}_{\mathcal{U}*} = \iota \circ \phi_{\mathcal{U}*}$ where ι is an isomorphism between the singular homology of $|N(\mathcal{U})|$ and the simplicial homology of $N(U)$. To see the second part, notice that $\bar{\phi}_{\mathcal{U}*}$ is a surjection by Theorem 8. Therefore, the classes $\bar{\phi}_{\mathcal{U}*}(z)$ where $\lambda(\mathcal{U}) \not> s(z)$ contain a basis for $H_1(N(\mathcal{U}))$. Hence they generate it.

Proof of (ii): Suppose on the contrary, there is a subsequence $\{\ell_1, \dots, \ell_t\} \subset \{\ell', \dots, g\}$ such that $\sum_{j=1}^t [\phi_{\mathcal{U}}(z_{\ell_j})] = 0$. Let $z = \sum_{j=1}^t \phi_{\mathcal{U}}(z_{\ell_j})$. Let γ be a 1-cycle in $N(\mathcal{U})$ so that $[z] = [|\gamma|]$ whose existence is guaranteed by Proposition 6. It must be the case that there is a 2-chain D in $N(\mathcal{U})$ so that $\partial D = \gamma$. Consider a triangle $t = \{u_{\alpha_1}, u_{\alpha_2}, u_{\alpha_3}\}$ contributing to D . Let $a'_i = \phi_{\mathcal{U}}^{-1}(u_{\alpha_i})$. Since t appears in $N(\mathcal{U})$, the covers $U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}$ containing a'_1, a'_2 , and a'_3 respectively have a common intersection in X . This also means that each of the paths $a'_1 \rightsquigarrow a'_2, a'_2 \rightsquigarrow a'_3, a'_3 \rightsquigarrow a'_1$ has size at most $2s_{max}(\mathcal{U})$. Then, $\rho(\partial t)$ is mapped to a 1-cycle in X of size at most $4s_{max}(\mathcal{U})$. It follows that $\rho(\partial D)$ can be written as a linear combination of cycles of size at most $4s_{max}(\mathcal{U})$. Each of the 1-cycles of size at most $4s_{max}(\mathcal{U})$ is generated by basis elements z_1, \dots, z_k where $s(z_k) \leq 4s_{max}(\mathcal{U})$. Therefore, the class of $z' = \phi_{\mathcal{U}}(\rho(\gamma))$ is generated by a linear combination of the basis elements whose preimages have size at most $4s_{max}(\mathcal{U})$. The class $[z']$ is same as the class $[|\gamma|]$ by Proposition 22. But, by assumption $[|\gamma|] = [z]$ is generated by a linear combination of the basis elements whose sizes are larger than $4s_{max}(\mathcal{U})$ reaching a contradiction. \square

4.2 H_1 -classes in Reeb space

In this section we prove an analogue of Theorem 21 for Reeb spaces, which to our knowledge is new. The Reeb space of a function $f : X \rightarrow Z$, denoted R_f , is the quotient of X under the equivalence relation $x \sim_f x'$ if and only if $f(x) = f(x')$ and there exists a continuous path $\gamma \in \Gamma_X(x, x')$ such that $f \circ \gamma$ is constant. The induced quotient map is denoted $q : X \rightarrow R_f$ which is of course surjective. We show that q_* at the homology level is also surjective for H_1 when the codomain Z of f is a metric space. In fact, we prove a stronger statement: only ‘vertical’ homology classes (classes with strictly positive size) survive in a Reeb space which extends the result of Dey and Wang [12] for Reeb graphs.

Let \mathcal{V} be a path-connected cover of R_f . This induces a pullback cover denoted $\mathcal{U} = \{U_\alpha\}_{\alpha \in A} = \{q^{-1}(V_\alpha)\}_{\alpha \in A}$ on X . Let $N(\mathcal{U})$ and $N(\mathcal{V})$ denote the corresponding nerve complexes of \mathcal{U} and \mathcal{V} respectively. It is easy to see that $N(\mathcal{U}) = N(\mathcal{V})$ because $U_\alpha \cap U_{\alpha'} \neq \emptyset$ if

and only if $V_\alpha \cap V_{\alpha'} \neq \emptyset$. There are nerve maps $\phi_{\mathcal{V}} : R_f \rightarrow |N(\mathcal{V})|$ and $\phi_{\mathcal{U}} : X \rightarrow |N(\mathcal{U})|$ so that the following holds:

Proposition 24. *Consider the sequence $X \xrightarrow{q} R_f(X) \xrightarrow{\phi_{\mathcal{U}}} |N(\mathcal{V})| = |N(\mathcal{U})|$. Then, $\phi_{\mathcal{U}} = \phi_{\mathcal{V}} \circ q$.*

Proof. Consider a partition of unity $\{\varphi_\alpha\}_{\alpha \in A}$ subordinate to $\mathcal{V} = \{V_\alpha\}_{\alpha \in A}$. Without loss of generality, one can assume \mathcal{V} to be locally finite because X is paracompact. Then, consider the partition of unity subordinate to $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ given by $\varphi'_\alpha(q^{-1}(x)) = \varphi_\alpha(x)$. Let $\phi_{\mathcal{V}}$ and $\phi_{\mathcal{U}}$ be the nerve maps corresponding to the partition of unity of φ_α and φ'_α respectively. Then, $\phi_{\mathcal{U}}(x) = \phi_{\mathcal{V}}(q(x))$ proving the claim. \square

Let the codomain of the function $f : X \rightarrow Z$ be a *metric space* (Z, d_Z) . We first impose a pseudometric on X induced by f ; the one-dimensional version of this pseudometric is similar to the one used in [1] for Reeb graphs. Recall that given two points $x, x' \in X$ we denote by $\Gamma_X(x, x')$ the set of all continuous paths $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = x$ and $\gamma(1) = x'$.

Definition 25. *We define a pseudometric d_f on X as follows: for $x, x' \in X$,*

$$d_f(x, x') := \inf_{\gamma \in \Gamma_X(x, x')} \text{diam}_Z(f \circ \gamma).$$

Proposition 26. *$d_f : X \times X \rightarrow \mathbb{R}_+$ is a pseudometric.*

Proof. Symmetry, non-negativity, and the fact that $d_f(x, x) = 0$ for all $x \in X$ are evident. We prove the triangle inequality. We will use the following claim whose proof we omit.

Claim 4.1. *For all $A, B \subseteq Z$ with $A \cap B \neq \emptyset$ we have $\text{diam}_Z(A \cup B) \leq \text{diam}_Z(A) + \text{diam}_Z(B)$.*

Assume $x, x', x'' \in X$ are such that $a = d_f(x, x')$ and $a' = d_f(x', x'')$. Fix any $\varepsilon > 0$. Choose $\gamma \in \Gamma_X(x, x')$ and $\gamma' \in \Gamma_X(x', x'')$ such that $\text{diam}_Z(f \circ \gamma) < a + \frac{\varepsilon}{2}$ and $\text{diam}_Z(f \circ \gamma') < a' + \frac{\varepsilon}{2}$. Now consider the curve $\gamma'' : [0, 1] \rightarrow X$ defined by concatenating γ and γ' so that $\gamma'' \in \Gamma_X(x, x'')$. Then, by the above claim, we have

$$d_f(x, x'') \leq \text{diam}_Z(f \circ \gamma'') = \text{diam}_Z(\{f \circ \gamma\} \cup \{f \circ \gamma'\}) \leq a + a' + \varepsilon.$$

The claim is obtained by letting $\varepsilon \rightarrow 0$. \square

Similar to X , we endow R_f with a distance \tilde{d}_f that descends via the map q : for any equivalence classes $r, r' \in R_f$, pick $x, x' \in X$ with $r = q(x)$ and $r' = q(x')$, then define

$$\tilde{d}_f(r, r') := d_f(x, x').$$

The definition does not depend on the representatives x and x' chosen. In this manner we obtain the pseudometric space (R_f, \tilde{d}_f) . Let z_1, \dots, z_g be a minimal generator basis of $H_1(X)$ defined with respect to the pseudometric d_f and $q : X \rightarrow R_f$ be the quotient map.

Theorem 27. *Let $\ell \in [1, g]$ be the smallest integer so that $s(z_\ell) \neq 0$. If no such ℓ exists, $H_1(R_f)$ is trivial, otherwise, $\{[q(z_i)]\}_{i=\ell, \dots, g}$ is a basis for $H_1(R_f)$.*

Proof. Consider the sequence $X \xrightarrow{q} R_f \xrightarrow{\phi_{\mathcal{V}}} |N(\mathcal{V})|$ where \mathcal{V} is a cover of R_f .

Claim 4.2. q_* is a surjection.

Proof. Let $z'_1, z'_2, \dots, z'_{g'}$ be a minimal generator basis of $H_1(R_f)$ of the metric space (R_f, \tilde{d}_f) . Observe that $s(z'_i) \neq 0$ for any $i \in [1, g']$ because otherwise we have a z'_j for some $j \in [1, g']$ whose any two distinct points $x, x' \in z'_j$ satisfy $f(q^{-1}(x)) = f(q^{-1}(x'))$ and $q^{-1}(x)$ and $q^{-1}(x')$ are path connected in X . This is impossible by the definition of R_f .

Without loss of generality, assume that \mathcal{V} is fine enough so that it satisfies $0 < s_{\max}(\mathcal{V}) \leq \delta$ where $\delta = \frac{1}{4} \min\{s(z'_i)\}$. Since $\delta > 0$ due to the observation in the previous paragraph, such a cover exists. Then, by applying Theorem 21(ii), we obtain that $[\phi_{\mathcal{V}}(z'_i)]_{i=1, \dots, g'}$ are linearly independent in $H_1(|N(\mathcal{V})|)$. It follows that $\phi_{\mathcal{V}*}$ is injective. It is surjective too by Proposition 7. Therefore, $\phi_{\mathcal{V}*}$ is an isomorphism.

Let \mathcal{U} be the pullback cover of \mathcal{V} . Then, we have $\phi_{\mathcal{U}*} = \phi_{\mathcal{V}*} \circ q_*$ (Proposition 24) where $\phi_{\mathcal{U}*}$ is a surjection and $\phi_{\mathcal{V}*}$ is an isomorphism. It follows that q_* is a surjection. \square

By the previous claim, $\{[q(z_i)]\}_{i=1, \dots, g}$ generate $H_1(R_f)$. First, assume that ℓ as stated in the theorem exists. Let the cover \mathcal{U} be fine enough so that $0 < s_{\max}(\mathcal{U}) \leq \delta$ where $\delta = \frac{1}{4} \min\{s(z_i) \mid s(z_i) \neq 0\}$. Then, by applying Theorem 21(ii), we obtain that $[\phi_{\mathcal{U}}(z_i)]_{i=\ell, \dots, g}$ are linearly independent in $H_1(|N(\mathcal{U})|) = H_1(|N(\mathcal{V})|)$. Since $[\phi_{\mathcal{U}}(z_i)] = [\phi_{\mathcal{V}} \circ q(z_i)]$ by Proposition 24, $\{[q(z_i)]\}_{i=\ell, \dots, g}$ are linearly independent in $H_1(R_f)$. But, $[q(z_i)] = 0$ for $s(z_i) = 0$ and $\{[q(z_i)]\}_{i=1, \dots, g}$ generate $H_1(R_f)$. Therefore, $\{[q(z_i)]\}_{i=\ell, \dots, g}$ is a basis. In the case when ℓ does not exist, we have $s(z_i) = 0$ for every $i \in [1, g]$. Then, $[q(z_i)] = 0$ for every i rendering $H_1(R_f)$ trivial. \square

4.3 Persistence of H_1 -classes in mapper and multiscale mapper

To apply the results for nerves in section 4.1 to mappers and multiscale mappers, the Lebesgue number of the pullback covers of X becomes important. The following observation in this respect is useful. Remember that the size of a subset in X and hence the cover elements are measured with respect to the pseudometric d_f .

Proposition 28. *Let \mathcal{U} be a cover for the codomain Z . Then, the pullback cover $f^*\mathcal{U}$ has Lebesgue number $\lambda(\mathcal{U})$.*

Proof. Let $X' \subseteq X$ be any path-connected subset where $s(X') \leq \lambda(\mathcal{U})$. Then, $f(X') \subseteq Z$ has a diameter at most $\lambda(\mathcal{U})$ by the definition of size. Therefore, by the definition of Lebesgue number, $f(X')$ is contained in a cover element $U \in \mathcal{U}$. Clearly, a path connected component of $f^{-1}(U)$ contains X' since f is assumed to be continuous. It follows that there is a cover element in $f^*\mathcal{U}$ that contains X' . Since X' was chosen as an arbitrary subset of size at most $\lambda(\mathcal{U})$, we have $\lambda(f^*\mathcal{U}) \geq \lambda(\mathcal{U})$. At the same time, it is straightforward from the definition of size that each cover element in $f^*\mathcal{U}$ has at most the size of U for any $U \in \mathcal{U}$. Therefore, $\lambda(f^*\mathcal{U}) \leq \lambda(\mathcal{U})$ establishing the equality as claimed. \square

Notice that the smallest size $s_{\min}(f^*\mathcal{U})$ of an element of the pullback cover can be arbitrarily small even if $s_{\min}(\mathcal{U})$ is not. However, the Lebesgue number of \mathcal{U} can be leveraged for the mapper due to the above Proposition.

Given a cover \mathcal{U} of Z , consider the mapper $N(f^*\mathcal{U})$. Let z_1, \dots, z_g be a set of minimal generator basis for $H_1(X)$ where the metric in question is d_f . Then, as a consequence of Theorem 21 we have:

Theorem 29.

- i* Let $\ell = g + 1$ if $\lambda(\mathcal{U}) > s(z_g)$. Otherwise, let $\ell \in [1, g]$ be the smallest integer so that $s(z_\ell) > \lambda(\mathcal{U})$. If $\ell \neq 1$, the class $\phi_{\mathcal{U}^*}[z_j] = 0$ for $j = 1, \dots, \ell - 1$. Moreover, if $\ell \neq g + 1$, the classes $\{\phi_{\mathcal{U}^*}[z_j]\}_{j=\ell, \dots, g}$ generate $H_1(N(f^*\mathcal{U}))$.
- ii* The classes $\{\phi_{\mathcal{U}^*}[z_j]\}_{j=\ell, \dots, g}$ are linearly independent where $s(z_{\ell'}) > 4s_{\max}(\mathcal{U})$.
- iii* Consider a H_1 -persistence module of a multiscale mapper induced by a tower of path connected covers:

$$H_1(N(f^*\mathcal{U}_{\varepsilon_0})) \xrightarrow{s_1^*} H_1(N(f^*\mathcal{U}_{\varepsilon_1})) \xrightarrow{s_2^*} \dots \xrightarrow{s_n^*} H_1(N(f^*\mathcal{U}_{\varepsilon_n})) \quad (3)$$

Let $\hat{s}_{i^*} = s_{i^*} \circ s_{(i-1)^*} \circ \dots \circ \bar{\phi}_{\mathcal{U}_{\varepsilon_0}^*}$. Then, the assertions in (i) and (ii) hold for $H_1(N(f^*\mathcal{U}_{\varepsilon_i}))$ with the map $\hat{s}_{i^*} : X \rightarrow N(f^*\mathcal{U}_{\varepsilon_i})$.

Remark 4.1 (Persistence diagram approximation.). *The persistence diagram of the H_1 -persistence module considered in Theorem 29(iii) contains points whose birth coordinates are exactly zero. This is because all connecting maps are surjective by (i) and thus every class is born only at the beginning. The death coordinate of a point that corresponds to a minimal basis generator of size s is in between the index ε_i and ε_j where $s \geq 4s_{\max}(\mathcal{U}_{\varepsilon_i})$ and $s \leq \lambda(\mathcal{U}_{\varepsilon_j})$ because of the assertions (i) and (ii) in Theorem 29. Assuming covers whose λ and s_{\max} values are within a constant factor of each other (such as the ones described in next subsection), we can conclude that a generator of size s dies at some point cs for some constant c . Therefore, by computing a minimal generator basis of $N(\mathcal{U}_{\varepsilon_0})$ and computing their sizes provide a 4-approximation to the persistence diagram of the multiscale mapper in the log scale.*

4.4 Two special covers and intrinsic Čech complex

We discuss two special covers, one can be effectively computed and the other one is relevant in the context of the intrinsic Čech complex of a metric space. We say a cover \mathcal{U} of a metric space (Y, d) is (α, β) -cover if $\alpha \leq \lambda(\mathcal{U})$ and $\beta \geq s_{\max}(\mathcal{U})$.

A $(\delta, 4\delta)$ -cover: Consider a δ -sample P of Y , that is, every metric ball $B(y, \delta)$, $y \in Y$, contains a point in P . Observe that the cover $\mathcal{U} = \{B(p, 2\delta)\}_{p \in P}$ is a $(\delta, 4\delta)$ -cover for Z . Clearly, $s_{\max}(\mathcal{U}) \leq 4\delta$. To determine $\lambda(\mathcal{U})$, consider any subset $Y' \subseteq Y$ with $s(Y') \leq \delta$. There is a $p \in P$ so that $d_Y(p, Y') \leq \delta$. Let y' be the furthest point in Y' from p . Then, $d_Y(p, y') \leq d_Y(p, Y) + \text{diam}(Y') \leq 2\delta$ establishing that $\lambda(\mathcal{U}) \geq \delta$.

A $(\delta, 2\delta)$ -cover: Consider the infinite cover \mathcal{U} of Y where $\mathcal{U} = \{B(y, \delta)\}_{y \in Y}$. These are the set of all metric balls of radius δ . Clearly, $s_{\max}(\mathcal{U}) \leq 2\delta$. Any subset $Y' \subseteq Y$ with $s(Y') \leq \delta$ is contained in a ball $B(y, \delta)$ where y is any point in Y' . This shows that $\lambda(\mathcal{U}) \geq \delta$. A consequence of this observation and Theorem 21 is that the intrinsic Čech complexes satisfy some interesting property.

Definition 30. Given a metric space (Y, d_Y) , its intrinsic Čech complex $C^\delta(Y)$ at scale δ is defined to be the nerve complex of the set of intrinsic δ -balls $\{B(y, \delta)\}_{y \in Y}$.

Observation 31. Let $C^\delta(Y)$ denote the intrinsic Čech complex of a metric space Y at scale δ . Let \mathcal{U} denote the corresponding possibly infinite cover of Y . Let z_1, \dots, z_g be a minimal generator basis for $H_1(Y)$. Then, $\{\bar{\phi}_{\mathcal{U}*}(z_i)\}_{i=\ell, \dots, g}$ generate $H_1(C^\delta(Y))$ if ℓ is the smallest integer with $s(z_\ell) > \delta$. Furthermore, $\{\bar{\phi}_{\mathcal{U}*}(z_i)\}_{i=\ell', \dots, g}$ are linearly independent if $s(z_{\ell'}) > 8\delta$.

5 Higher dimensional homology groups

We have already observed that the surjectivity of the map $\phi_{\mathcal{U}*} : H_1(X) \rightarrow H_1(|N(\mathcal{U})|)$ in one dimensional homology does not extend to higher dimensional homology groups. This means that we cannot hope for analogues to Theorem 21(i) and Theorem 29 to hold for higher dimensional homology groups. However, under the assumption that $f : X \rightarrow Z$ is a continuous map from a compact space to a metric space, we can provide some characterization of the persistent diagrams of the mapper and the multiscale mapper as follows:

- We define a metric d_δ on the vertex set P_δ of $N(\mathcal{U})$ where $s_{\max}(\mathcal{U}) \leq \delta$ and then show that the Gromov-Hausdorff distance between the metric spaces (P_δ, d_δ) and (R_f, \tilde{d}_f) is at most 5δ . The same proof also applies if we replace (R_f, \tilde{d}_f) with the pseudometric space (X, d_f) .
- Previous result implies that the persistence diagrams of the intrinsic Čech complex of the metric space (X, d_f) and that of the metric space (P_δ, d_δ) have a bottleneck distance of $O(\delta)$. This further implies that the persistence diagram of the mapper structure $N(\mathcal{U})$ (approximated with the metric space (P_δ, d_δ)) is close to that of the intrinsic Čech complex of the pseudometric space (X, d_f) ; see Section 5.2.1.
- We show that the intrinsic Čech complexes of (X, d_f) interleave with $\text{MM}(\mathcal{U}, f)$ thus connecting their persistence diagrams. See Section 5.2.2.
- It follows that the persistence diagrams of the multiscale mapper $\text{MM}(\mathcal{U}, f)$ and (P_δ, d_δ) are close, both being close to that of (X, d_f) . This shows that the multiscale mapper encodes similar information as the mapper under an appropriate map-induced metric.

5.1 Gromov-Hausdorff distance between Mapper and the Reeb space

5.1.1 Mapper as a finite metric space

We have already shown how to equip the Reeb space R_f with a distance \tilde{d}_f .

Consider a cover \mathcal{U}_δ of Z whose all cover elements have size at most δ , that is, $\mathcal{U}_\delta = \{U_\alpha, \alpha \in A, s(U_\alpha) \leq \delta\}$. For a continuous map $f : X \rightarrow Z$ consider now the pullback cover $\mathcal{V}_\delta = f^*\mathcal{U}_\delta$ of X consisting of elements $\{V_{\alpha,i}, i \in I_\alpha \text{ and } \alpha \in A\}$. We choose an arbitrary but distinct point $z_\alpha \in U_\alpha$ for every element $U_\alpha \in \mathcal{U}_\delta$.

Consider now the nerve $M_\delta = N(\mathcal{V}_\delta)$, and let P_δ denote the vertex set of M_δ ; we will denote its points by $v_{\alpha,i}$ which corresponds to the element $V_{\alpha,i}$. Denote by E_δ the edge set of M_δ .

Define the vertex function $f_\delta : P_\delta \rightarrow Z$ as follows: $f_\delta(v_{\alpha,i}) := z_\alpha$ for each $v_{\alpha,i} \in P_\delta$. Consider the metric $d_\delta : P_\delta \times P_\delta \rightarrow \mathbb{R}_+$ given by

$$d_\delta(v, v') := \min \{ \text{diam}_Z(\{f_\delta(v_\ell)\}_{\ell=0}^n), \text{ where } v_0 = v, v_n = v', (v_k, v_{k+1}) \in E_\delta \text{ for all } k \}$$

for any $v, v' \in P_\delta$. We thus form the finite metric space (P_δ, d_δ) .

Remark 5.1. *Verifying that d_δ is indeed a metric requires checking that $d_\delta(v, v') = 0$ implies that $v = v'$.³ If $d_\delta(v, v') = 0$ then there exist $v = v_0, \dots, v_n = v'$ in P_δ and $z_* \in Z$ such that $(v_k, v_{k+1}) \in E_\delta$ for all k such that $f_\delta(v_k) = z_*$ for all k . If we write $v_k = v_{\alpha_k, i_k}$ for $i_k \in I_{\alpha_k}$ then this means that $z_* = f_\delta(v_{\alpha_k, i_k}) = z_{\alpha_k}$ for all k . This means that the elements $\{V_{\alpha_k, i_k}, k = 0, \dots, n\}$ of the pullback cover are all different path connected components of the set $f^{-1}(U_{\alpha_k})$. This means that one cannot have $(v_k, v_{k+1}) \in E_\delta$ unless $v_0 = v_1 = \dots, v_n$ implying that $v = v'$.*

We now construct a map $p_\delta : X \rightarrow P_\delta$. In order to do this consider the set of indices $B = \{(\alpha, i), \alpha \in A, i \in I_\alpha\}$ into elements of the cover $\mathcal{V}_\delta = f^*\mathcal{U}_\delta$. Choose any total order $>_A$ on A , and then declare that $(\alpha, i) > (\alpha', i')$ whenever it holds (1) $\alpha >_A \alpha'$, or (2) in case $\alpha = \alpha', i > i'$. For any $x \in X$ let $p_\delta(x) := v_{\alpha, i}$ where $(\alpha, i) = \min\{\beta \in B \mid x \in V_\beta\}$.

Notice that p_δ is not necessarily a surjection. Since our goal is to define a correspondence between X and P_δ , for every $V_{\alpha, i} \in \mathcal{V}_\delta$ we choose an arbitrary point $x_{\alpha, i} \in V_{\alpha, i}$ and associate it with the vertex $v_{\alpha, i}$.

5.1.2 A bound on the Gromov-Hausdorff distance

The proof of the following theorem extends to (X, d_f) almost verbatim.

Theorem 32. *Under the conditions above,*

$$d_{GH}((R_f, \tilde{d}_f), (P_\delta, d_\delta)) \leq 5\delta.$$

Proof. Consider the correspondence S between R_f and P_δ defined by $S := \{(q(x), p_\delta(x)), x \in X\} \cup \{q(x_{\alpha, i}), v_{\alpha, i}\}$. That S is indeed a correspondence follows from the fact that $q : X \rightarrow R_f$ is a surjection and the second factor in S covers all vertices in P_δ .

Claim 5.1. *For all $x, x' \in X$ one has*

$$\tilde{d}_f(q(x), q(x')) - \delta \leq d_\delta(p_\delta(x), p_\delta(x')) \leq \tilde{d}_f(q(x), q(x')) + \delta.$$

Claim 5.2. *For all $x, x_{\alpha, i} \in X$ one has*

$$\tilde{d}_f(q(x), q(x_{\alpha, i})) - 3\delta \leq d_\delta(p_\delta(x), v_{\alpha, i}) \leq \tilde{d}_f(q(x), q(x_{\alpha, i})) + 3\delta$$

Claim 5.3. *For all $x_{\alpha, i}, x_{\alpha', i'} \in X$ one has*

$$\tilde{d}_f(q(x_{\alpha, i}), q(x_{\alpha', i'})) - 5\delta \leq d_\delta(v_{\alpha, i}, v_{\alpha', i'}) \leq \tilde{d}_f(q(x_{\alpha, i}), q(x_{\alpha', i'})) + 5\delta$$

³The triangle inequality is clear.

Combining the three claims above we obtain that

$$\text{dis}(S) = \sup_{x,x' \in X, y \in S(x), y' \in S(x')} |\tilde{d}_f(q(x), q(x')) - d_\delta(y, y')| \leq 5\delta$$

thus finishing the proof. \square

Proof of Claim 5.1. We prove the upper bound. The proof for the lower bound is similar. Assume that $\tilde{d}_f(q(x), q(x')) < \eta$ for some $\eta > 0$ and let $\gamma \in \Gamma_X(x, x')$ be s.t. $\text{diam}_Z(f \circ \gamma) \leq \eta$. Consider the set of vertices $Q := \{p_\delta(\gamma(t)), t \in [0, 1]\} \subset P_\delta$. This set consists of a finite sequence of vertices v_{α_ℓ, i_ℓ} for $\ell = 0, 1, \dots, N$, for some positive integer N . Notice that $f_\delta(Q) = \{z_{\alpha_\ell}, \ell = 0, 1, \dots, N\}$ and by construction we can assume that $(v_{\alpha_\ell, i_\ell}, v_{\alpha_{\ell+1}, i_{\ell+1}}) \in E_\delta$ for each ℓ .

Now, for each $\ell \in \{0, \dots, N\}$ there exists $t_\ell \in [0, 1]$ such that $\gamma(t_\ell) \in V_{\alpha_\ell, i_\ell}$, which means that $f(\gamma(t_\ell)) \in U_{\alpha_\ell}$. But $z_{\alpha_\ell} \in U_{\alpha_\ell}$ so that then $f(Q) \subseteq \bigcup_{\ell=0}^N U_{\alpha_\ell}$. At the same time, $\bigcup_{t \in [0, 1]} (f(\gamma(t))) \subseteq \bigcup_{\ell=0}^N U_{\alpha_\ell}$. Hence,

$$\delta + \eta \geq \delta + \text{diam}_Z(f \circ \gamma) \geq \text{diam}_Z(f(Q)) \geq d_\delta(p_\delta(x), p_\delta(x')).$$

The proof of the upper bound follows by letting $\eta \rightarrow \tilde{d}_f(q(x), q(x'))$. \square

To prove Claims 5.2 and 5.3, we first observe the following.

Observation 33. *For each $x_{\alpha, i}$ one has $d_\delta(p_\delta(x_{\alpha, i}), v_{\alpha, i}) \leq 2\delta$.*

Proof. Let $p_\delta(x_{\alpha, i}) = v_{\alpha', i'}$. This means that $x_{\alpha, i} \in V_{\alpha, i} \cap V_{\alpha', i'}$. Therefore, $(v_{\alpha, i}, v_{\alpha', i'}) \in E_\delta$ is an edge. Since $V_{\alpha, i}$ and $V_{\alpha', i'}$ intersects, so does U_α and $U_{\alpha'}$. Therefore, $d_Z(f_\delta(v_{\alpha, i}), f_\delta(v_{\alpha', i'})) = d_Z(z_\alpha, z_{\alpha'}) \leq 2\delta$ establishing that $\text{diam}_Z(\{f_\delta(v_{\alpha, i}), f_\delta(v_{\alpha', i'})\}) \leq 2\delta$. \square

Proof of Claim 5.2. Again, we prove only the upper bound since the lower bound proof is similar. We have (by Claim 5.1) $\tilde{d}_f(q(x), q(x_{\alpha, i})) \leq d_\delta(p_\delta(x), p_\delta(x_{\alpha, i})) + \delta$ The righthand side is at most $d_\delta(p_\delta(x), v_{\alpha, i}) + d_\delta(p_\delta(x_{\alpha, i}), v_{\alpha, i}) + \delta$ by triangular inequality. Applying Observation 33, we get $d_\delta(p_\delta(x), v_{\alpha, i}) + d_\delta(p_\delta(x_{\alpha, i}), v_{\alpha, i}) + 2\delta \leq d_\delta(p_\delta(x), v_{\alpha, i}) + 3\delta$ proving the claim. \square

Proof of Claim 5.3. We have

$$\begin{aligned} \tilde{d}_f(q(x_{\alpha, i}), q(x_{\alpha', i'})) &\leq d_\delta(p_\delta(x_{\alpha, i}), p_\delta(x_{\alpha', i'})) + \delta \\ &\leq d_\delta(p_\delta(x_{\alpha, i}), v_{\alpha, i}) + d_\delta(v_{\alpha, i}, p_\delta(x_{\alpha', i'})) + \delta \\ &\leq d_\delta(v_{\alpha, i}, p_\delta(x_{\alpha', i'})) + 3\delta \\ &\leq d_\delta(v_{\alpha, i}, v_{\alpha', i'}) + 5\delta \end{aligned}$$

The lower bound can be shown similarly. \square

5.2 Interleaving of persistent homology groups

5.2.1 Intrinsic Čech complex filtrations for $(N(\mathcal{U}), d_\delta)$ and for (X, d_f)

Definition 34 (Intrinsic Čech filtration). *The intrinsic Čech filtration of the metric space (Y, d_Y) is*

$$\mathfrak{C}(Y) = \{C^r(Y) \subseteq C^{r'}(Y)\}_{0 < r < r'}.$$

The intrinsic Čech filtration at resolution s is defined as $\mathfrak{C}_s(Y) = \{C^r(Y) \subseteq C^{r'}(Y)\}_{s \leq r < r'}$.

Whenever (Y, d_Y) is totally bounded, the persistence modules induced by taking homology of this intrinsic Čech filtration become q-tame [8]. This implies that one may define its persistence diagram $\text{Dg } \mathfrak{C}(Y)$ which provides one way to summarize the topological information of the space Y through the lens of its metric structure d_Y .

We prove that the pseudometric space (X, d_f) is totally bounded. This requires us to show that for any $\varepsilon > 0$ there is a finite subset of $P \subseteq X$ so that open balls centered at points in P with radii ε cover X . Recall that we have assumed that X is a compact topological space, that (Z, d_Z) is a metric space, and that $f : X \rightarrow Z$ is a continuous map. Consider a cover \mathcal{U} of Z where each cover element is a ball of radius most $\varepsilon/2$ around a point in Z . Then, the pullback cover $f^*\mathcal{U}$ of X has all elements with diameter at most ε in the metric d_f . Since X is compact, a finite sub-cover of $f^*\mathcal{U}$ still covers X . A finite set P consisting of one arbitrary point in each element of this finite sub-cover is such that the union of d_f -balls of radius ε around points in P covers X . Since $\varepsilon > 0$ was arbitrary, (X, d_f) is totally bounded.

Consider the mapper $N(f^*\mathcal{U})$ w.r.t a cover \mathcal{U} of the codomain Z . We can equip its vertex set, denoted by P_δ , with a metric structure (P_δ, d_δ) , where δ is an upper bound on the diameter of each element in \mathcal{U} . Hence we can view the persistence diagram $\text{Dg } \mathfrak{C}(P_\delta)$ w.r.t. the metric d_δ as a summary of the mapper $N(f^*\mathcal{U})$. Using the Gromov-Hausdorff distance between the metric spaces (P_δ, d_δ) and (X, d_f) , we relate this persistent summary to the persistence diagram $\text{Dg } \mathfrak{C}(X)$ induced by the intrinsic Čech filtration of (X, d_f) . Specifically, we show that $d_{GH}((P_\delta, d_\delta), (X, d_f)) \leq 5\delta$. With (X, d_f) being totally bounded, by results of [8]⁴, it follows that the bottleneck-distance between the two resulting persistence diagrams satisfies:

$$d_B(\text{Dg } \mathfrak{C}(P_\delta), \text{Dg } \mathfrak{C}(X)) \leq 2 * 5\delta = 10\delta. \quad (4)$$

5.2.2 $\text{MM}(\mathfrak{W}, f)$ for a tower of covers \mathfrak{W}

Above we discussed the information encoded in a certain persistence diagram summary of a single Mapper structure. We now consider the persistent homology of multiscale mappers. Given any tower of covers (TOC) \mathfrak{W} of the co-domain Z , by applying the homology functor to its multiscale mapper $\text{MM}(\mathfrak{W}, f)$, we obtain a persistent module, and we can thus discuss the persistent homology induced by a tower of covers \mathfrak{W} . However, as discussed in [11], this persistent module is not necessarily stable under perturbations (of e.g the map f) for general TOCs. To address this issue, Dey et al. introduced a special family of the so-called

⁴Although d_f is a pseudo-metric, the bound on Gromov-Hausdorff distance still implies that the two intrinsic Čech filtrations $\mathfrak{C}(P_\delta)$ and $\mathfrak{C}(X)$ are interleaved. Now, since (X, d_f) is totally bounded, we can apply results of [8] in our setting.

(c,s)-good TOC in [11], which is natural and still general. Below we provide an equivalent definition of the (c,s)-good TOC based on the Lebesgue number of covers.

Definition 35 ((c,s)-good TOC). *Give a tower of covers $\mathfrak{U} = \{\mathcal{U}_\varepsilon\}_{\varepsilon \geq s}$, we say that it is (c,s)-good TOC if for any $\varepsilon \geq s$, we have that (i) $s_{\max}(\mathcal{U}_\varepsilon) \leq \varepsilon$ and (ii) $\lambda(\mathcal{U}_{c\varepsilon}) \geq \varepsilon$.*

As an example, the TOC $\mathfrak{U} = \{\mathcal{U}_\varepsilon\}_{\varepsilon \geq s}$ with $\mathcal{U}_\varepsilon := \{B_{\varepsilon/2}(z) \mid z \in Z\}$ is an (2,s)-good TOC of the co-domain Z .

We now characterize the persistent homology of multiscale mappers induced by (c,s)-good TOCs. Connecting these persistence modules is achieved via the interleaving of towers of simplicial complexes originally introduced in [6]. Below we include the slightly generalized version of the definition from [11].

Definition 36 (Interleaving of simplicial towers, [11]). *Let $\mathfrak{S} = \{\mathcal{S}_\varepsilon \xrightarrow{s_{\varepsilon,\varepsilon'}} \mathcal{S}_{\varepsilon'}\}_{r \leq \varepsilon \leq \varepsilon'}$ and $\mathfrak{T} = \{\mathcal{T}_\varepsilon \xrightarrow{t_{\varepsilon,\varepsilon'}} \mathcal{T}_{\varepsilon'}\}_{r \leq \varepsilon \leq \varepsilon'}$ be two towers of simplicial complexes where $\text{res}(\mathfrak{S}) = \text{res}(\mathfrak{T}) = r$. For some $c \geq 0$, we say that they are c-interleaved if for each $\varepsilon \geq r$ one can find simplicial maps $\varphi_\varepsilon : \mathcal{S}_\varepsilon \rightarrow \mathcal{T}_{\varepsilon+c}$ and $\psi_\varepsilon : \mathcal{T}_\varepsilon \rightarrow \mathcal{S}_{\varepsilon+c}$ so that:*

- (i) for all $\varepsilon \geq r$, $\psi_{\varepsilon+c} \circ \varphi_\varepsilon$ and $s_{\varepsilon,\varepsilon+2c}$ are contiguous,
- (ii) for all $\varepsilon \geq r$, $\varphi_{\varepsilon+\eta} \circ \psi_\varepsilon$ and $t_{\varepsilon,\varepsilon+2c}$ are contiguous,
- (iii) for all $\varepsilon' \geq \varepsilon \geq r$, $\varphi_{\varepsilon'} \circ s_{\varepsilon,\varepsilon'}$ and $t_{\varepsilon+c,\varepsilon'+c} \circ \varphi_\varepsilon$ are contiguous,
- (iv) for all $\varepsilon' \geq \varepsilon \geq r$, $s_{\varepsilon+c,\varepsilon'+c} \circ \psi_\varepsilon$ and $\psi_{\varepsilon'} \circ t_{\varepsilon,\varepsilon'}$ are contiguous.

Analogously, if we replace the operator ‘+’ by the multiplication ‘.’ in the above definition, then we say that \mathfrak{S} and \mathfrak{T} are c-multiplicatively interleaved.

Furthermore, all the simplicial towers that we will encounter here will be those induced by taking the nerve of some tower of covers (TOCs). It turns out that the interleaving of such tower of nerve complexes can be identified via interleaving of their corresponding tower of covers, which is much easier to verify. More precisely,

Definition 37 ((Multiplicative) Interleaving of towers of covers, [11]). *Let $\mathfrak{V} = \{\mathcal{V}_\varepsilon\}$ and $\mathfrak{W} = \{\mathcal{W}_{\varepsilon'}\}$ be two towers of covers of a topological space X such that $\text{res}(\mathfrak{V}) = \text{res}(\mathfrak{W}) = r$. Given $\eta \geq 0$, we say that \mathfrak{V} and \mathfrak{W} are η -multiplicatively interleaved if one can find maps of covers $\zeta_\varepsilon : \mathcal{V}_\varepsilon \rightarrow \mathcal{W}_{\eta\varepsilon}$ and $\xi_{\varepsilon'} : \mathcal{W}_{\varepsilon'} \rightarrow \mathcal{V}_{\eta\varepsilon'}$ for all $\varepsilon, \varepsilon' \geq r$.*

The following two results of [11] connect interleaving TOCs with the interleaving of their induced tower of nerve complexes and multiscale mappers ⁵.

Proposition 38 (Proposition 4.2 of [11]). *Let \mathfrak{U} and \mathfrak{V} be two η -(multiplicatively) interleaved towers of covers of X with $\text{res}(\mathfrak{U}) = \text{res}(\mathfrak{V})$. Then, $N(\mathfrak{U})$ and $N(\mathfrak{V})$ are also η -(multiplicatively) interleaved.*

⁵These propositions are proven in [11] for the additive version of interleaving; but the same proofs hold for the multiplicative interleaving case.

Proposition 39 (Proposition 4.1 of [11]). *Let $f : X \rightarrow Z$ be a continuous function and \mathfrak{U} and \mathfrak{V} be two η -(multiplicatively) interleaved tower of covers of Z . Then, $f^*(\mathfrak{U})$ and $f^*(\mathfrak{V})$ are also η -(multiplicatively) interleaved.*

By Proposition 38, this implies that the resulting multiscale mappers $\text{MM}(\mathfrak{U}, f)$ and $\text{MM}(\mathfrak{V}, f)$ are also η -(multiplicatively) interleaved.

Our main results of this section are the following . First, Theorem 40 states that the multiscale-mappers induced by any two (c, s) -good towers of covers interleave with each other, implying that their respective persistence diagrams are also close under the bottleneck distance. From this point of view, the persistence diagrams induced by any two (c, s) -good TOCs contain roughly the same information. Next in Theorem 41, we show that the multiscale mapper induced by any (c, s) -good TOC interleaves (at the homology level) with the intrinsic Čech filtration of (X, d_f) , thereby implying that the persistence diagram of the multiscale mapper w.r.t. any (c, s) -good TOC is close to that of the intrinsic Čech filtration of (X, d_f) under the bottleneck distance.

Theorem 40. *Given a map $f : X \rightarrow Z$, let $\mathfrak{V} = \{\mathcal{V}_\varepsilon \xrightarrow{v_{\varepsilon, \varepsilon'}} \mathcal{V}_{\varepsilon'}\}_{\varepsilon \leq \varepsilon'}$ and $\mathfrak{W} = \{\mathcal{W}_\varepsilon \xrightarrow{w_{\varepsilon, \varepsilon'}} \mathcal{W}_{\varepsilon'}\}_{\varepsilon \leq \varepsilon'}$ be two (c, s) -good tower of covers of Z . Then the corresponding multiscale mappers $\text{MM}(\mathfrak{V}, f)$ and $\text{MM}(\mathfrak{W}, f)$ are c -multiplicatively interleaved.*

Proof. First, we make the following observation.

Claim 5.4. *Any two (c, s) -good TOCs \mathfrak{V} and \mathfrak{W} are c -multiplicatively interleaved.*

Proof. It follows easily from the definitions of (c, s) -good TOC. Specifically, first we construct $\zeta_\varepsilon : \mathcal{V}_\varepsilon \rightarrow \mathcal{W}_{c\varepsilon}$. For any $V \in \mathcal{V}_\varepsilon$, we have that $\text{diam}(V) \leq \varepsilon$. Furthermore, since \mathfrak{W} is (c, s) -good, there exists $W \in \mathcal{W}_{c\varepsilon}$ such that $V \subseteq W$. Set $\zeta_\varepsilon(V) = W$; if there are multiple choice of W , we can choose an arbitrary one. We can construct $\xi_{\varepsilon'} : \mathcal{W}_{\varepsilon'} \rightarrow \mathcal{V}_{c\varepsilon'}$ in a symmetric manner, and the claim then follows. \square

This, combined with Propositions 39 and 38, prove the theorem. \square

Recall the definition of intrinsic Čech complex filtration $\mathfrak{C} - s(Y)$ at resolution s for a metric space (Y, d_Y) in Def 34.

Theorem 41. *Let $\mathfrak{C}_s(X)$ be the intrinsic Čech filtration of (X, d_f) starting with resolution s . Let $\mathfrak{U} = \{\mathcal{U}_\varepsilon \xrightarrow{u_{\varepsilon, \varepsilon'}} \mathcal{U}_{\varepsilon'}\}_{s \leq \varepsilon \leq \varepsilon'}$ be a (c, s) -good TOC of the compact connected metric space Z . Then the multiscale mapper $\text{MM}(\mathfrak{U}, f)$ and $\mathfrak{C}_s(X)$ are $2c$ -multiplicatively interleaved.*

Proof. Let $D_\varepsilon := \{B_\varepsilon(x) \mid x \in X\}$ be the infinite cover of X consisting of all ε -intrinsic balls in X . Obviously, $C^\varepsilon(X)$ is the nerve complex induced by the cover D_ε , for each $\varepsilon > 0$. Let $\mathfrak{D} = \{D_\varepsilon \xrightarrow{t_{\varepsilon, \varepsilon'}} D_{\varepsilon'}\}_{s \leq \varepsilon \leq \varepsilon'}$ be the corresponding tower of covers of X , where $t_{\varepsilon, \varepsilon'}$ sends $B_\varepsilon(x) \in D_\varepsilon$ to $B_{\varepsilon'}(x) \in D_{\varepsilon'}$. Obviously, the tower of the nerve complexes for D_ε s give rise to $\mathfrak{C}_s(X)$.

On the other hand, let $W_\varepsilon = f^*\mathcal{U}_\varepsilon$ be the pull-back cover of X induced by \mathcal{U}_ε via f , and $\mathfrak{W} = f^*\mathfrak{U}$ is the pull-back tower of cover of X induced by the TOC \mathfrak{U} of Z . By definition,

we know that the multiscale mapper $\text{MM}(\mathfrak{U}, f) = \{M_\varepsilon \xrightarrow{s_{\varepsilon, \varepsilon'}} M_{\varepsilon'}\}_{s \leq \varepsilon \leq \varepsilon'}$ where M_ε is the nerve complex of the cover W_ε .

In what follows, we will argue that the two TOCs \mathfrak{D} and \mathfrak{W} are $2c$ -multiplicatively interleaved. By Proposition 38, this then proves the theorem.

First, we show that there is a map of covers $\zeta_\varepsilon : D_\varepsilon \rightarrow W_{2c\varepsilon}$ for each $\varepsilon \geq s$ defined as follows.

Take any intrinsic ball $B_{\varepsilon, d_f}(x) \in D_\varepsilon$ for some $x \in X$. Consider the image $f(B_\varepsilon(x)) \subseteq Z$. Recall that the covering metric $d_f(x_1, x_2)$ on X is defined by the minimum diameter of the image of any path ρ connecting x_1 to x_2 in X ; that is, $d_f(x_1, x_2) = \inf_{\rho: x_1 \rightsquigarrow x_2} \text{diam}(f(\rho))$. Thus $d_Z(f(x_1), f(x_2)) \leq d_f(x_1, x_2)$. We then have that for any $x_1, x_2 \in B_\varepsilon(x)$,

$$d_Z(f(x_1), f(x_2)) \leq d_Z(f(x_1), f(x)) + d_Z(f(x), f(x_2)) \leq d_f(x_1, x) + d_f(x_2, x) \leq 2\varepsilon.$$

This implies that $\text{diam}(f(B_\varepsilon(x))) \leq 2\varepsilon$. Since \mathfrak{U} is a (c, s) -good TOC, it then follows that there exists $U_x \in \mathcal{U}_{2c\varepsilon}$ such that $f(B_\varepsilon(x)) \subseteq U$ (if there are multiple elements contains $f(B_\varepsilon(x))$, we can choose an arbitrary one as U_x). This means that $B_\varepsilon(x)$ is contained within one of the connected component, say W_x in $\text{cc}(f^{-1}(U_x))$. We simply set $\zeta_\varepsilon(B_\varepsilon(x)) = W_x \in W_{2c\varepsilon}$.

Finally we show that there is a map of covers $\xi_\varepsilon : W_\varepsilon \rightarrow D_\varepsilon \xrightarrow{t_{\varepsilon, 2\varepsilon}} D_{2c\varepsilon}$. To this end, consider any set $V \in W_\varepsilon$; by definition, there exists some $U \in \mathcal{U}_\varepsilon$ such that $V \in \text{cc}(f^*(U))$. Note, $f(V) \subseteq U$ and $\text{diam}(U) \leq s_{\max}(\mathcal{U}_\varepsilon) \leq \varepsilon$. It then follows from the definition of the metric d_f that for any point x from V , we have that $V \subseteq B_{\varepsilon, d_f}(x)$. We simply set $\xi_\varepsilon(V) = B_\varepsilon(x)$. This completes the proof that the two TOCs \mathfrak{D} and \mathfrak{W} are $2c$ -multiplicatively interleaved. The theorem then follows this and Proposition 38. \square

Finally, given a persistence diagram Dg , we denote its *log-scaled version* Dg_{\log} to be the diagram consisting of the set of points $\{(\log x, \log y) \mid (x, y) \in \text{Dg}\}$. Since interleaving towers of simplicial complexes induce interleaving persistent modules, using results of [6, 7], we have the following corollary.

Corollary 42. *Given a continuous map $f : X \rightarrow Z$ and a (c, s) -good TOC \mathfrak{U} of Z , let $\text{Dg}_{\log} \text{MM}(\mathfrak{U}, f)$ and $\text{Dg}_{\log} \mathfrak{C}_s$ denote the log-scaled persistence diagram of the persistence modules induced by $\text{MM}(\mathfrak{U}, f)$ and by the intrinsic Čech filtration \mathfrak{C}_s of (X, d_f) respectively. We have that*

$$d_B(\text{Dg}_{\log} \text{MM}(\mathfrak{U}, f), \text{Dg}_{\log} \mathfrak{C}_s) \leq 2c.$$

6 Concluding remarks

In this paper, we present some studies on the topological information encoded in Nerves, Reeb spaces, mappers and multiscale mappers, where the latter two structures are constructed based on nerves. Currently, the characterization for the H_1 -homology for the Nerve complex is much stronger than for higher dimensions. In particular, we showed that for a path-connected cover \mathcal{U} , there is a surjection from the domain $H_1(X)$ to $H_1(N(\mathcal{U}))$. While this does not hold for higher dimensional cases (as Figure 1 demonstrates), we wonder if similar surjection holds under additional conditions on the input cover such as the ones

used by Björner [3] for homotopy groups. Along that line, we ask: if for any $k \geq 0$, t -wise intersections of cover elements for all $t > 0$ have trivial reduced homology groups for all dimensions up to $k - t$, then does the nerve map induce a surjection for the k -dimensional homology? We have answered it affirmatively for $k = 1$.

We also remark that it is possible to carry out most of our arguments using the language of category theory (see e.g. [23] on this view for the mapper structure). We choose not to take this route and explain the results with more elementary expositions.

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