Simplex Algorithm

Advanced Algorithms (CSE 794)
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April 30, 2009

1 Introduction

The simplex method is a method for solving a linear program created by George Dantzig. While its running time is not polynomial in the worst case, it performs quite well in practice. The method makes use of the concept of the simplex which is a polytope of N+1 vertices in N dimensions. The polytope has the property that every point on a line drawn between two vertices will lie within the polytope. This region inside the polytope is known as the feasible solution. The solution that maximizes the set of linear inequalities will be a vertex on the polytope.

2 Simplex Algorithm Example

Before formally presenting the simplex algorithm, we will examine an illustrative example. We will solve the following linear program.

$$\text{maximize } 3x_1 + x_2 + 2x_3 \tag{1a}$$

subject to

$$x_1 + x_2 + 3x_3 \leq 30 \tag{1b}$$
$$2x_1 + 2x_2 + 5x_3 \leq 24 \tag{1c}$$
$$4x_1 + x_2 + 2x_3 \leq 36 \tag{1d}$$
$$x_1, x_2, x_3 \geq 0 \tag{1e}$$

In order to solve this program using the simplex algorithm, we must first convert it to slack form. The procedure for transforming a given linear program into slack form was discussed in the previous lecture. By introducing the variables x_4, x_5, and x_6, we produce the slack form below.

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$$z = 3x_1 + x_2 + 2x_3 \tag{2a}$$
$$x_4 = 30 - x_1 - x_2 - 3x_3 \tag{2b}$$
$$x_5 = 24 - 2x_1 - 2x_2 - 5x_3 \tag{2c}$$
$$x_6 = 36 - 4x_1 - x_2 - 2x_3 \tag{2d}$$

At each step we can also obtain a **basic solution**. We do this by setting all non-basic variables to 0. Initially the basic solution gives us a value of 0. This should increase at every step.
While this slack form is useful for applying the algorithm, we will need a more succinct format for when we eventually formalize the algorithm. This format is presented below and will be shown for each step.

$$B = \{4,5,6\} N = \{1,2,3\}$$
$$c = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} b = \begin{pmatrix} 30 \\ 24 \\ 36 \end{pmatrix}$$
$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 5 \\ 4 & 1 & 2 \end{pmatrix} v = 0$$

Now we will choose one of the coefficients from the equation we are maximizing. We will choose x_1 and increase it by the maximum amount that will still ensure that every basic variable is non-negative. We rewrite equation 2d in terms of x_1. Next, replace all values of x_1 and obtain the new slack form.

$$z = 27 + \frac{x_2}{4} + \frac{x_3}{2} - \frac{3x_6}{4} \tag{3a}$$
$$x_1 = 9 - \frac{x_2}{4} - \frac{x_3}{2} - \frac{x_6}{4} \tag{3b}$$
$$x_4 = 21 - \frac{3x_2}{4} - \frac{5x_3}{2} + \frac{x_6}{4} \tag{3c}$$
$$x_5 = 6 - \frac{3x_2}{2} - 4x_3 + \frac{x_6}{2} \tag{3d}$$

$$B = \{1,4,5\} N = \{2,3,6\}$$
$$c = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ -\frac{3}{4} \end{pmatrix} b = \begin{pmatrix} 9 \\ 21 \\ 6 \end{pmatrix}$$
$$A = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{3}{4} & \frac{5}{2} & -\frac{1}{4} \\ \frac{3}{2} & 4 & \frac{1}{2} \end{pmatrix} v = 27$$

Now only two variables are available for us to select since we can only use the positive basic variables. To produce the next slack form we will rewrite equation 3d in terms of x_3.

$$z = \frac{111}{4} + \frac{x_2}{16} - \frac{x_5}{8} - \frac{11x_6}{16} \tag{4a}$$
$$x_1 = \frac{33}{4} - \frac{x_2}{16} + \frac{x_5}{8} - \frac{5x_6}{16} \tag{4b}$$
$$x_3 = \frac{3}{2} - \frac{3x_2}{8} + \frac{x_5}{4} + \frac{x_6}{8} \tag{4c}$$
$$x_4 = \frac{69}{4} + \frac{3x_2}{16} + \frac{5x_5}{8} - \frac{x_6}{16} \tag{4d}$$

$$B = \{1, 3, 4\} N = \{2, 5, 6\}$$
$$c = \begin{pmatrix} \frac{1}{16} \\ -\frac{1}{8} \\ -\frac{11}{16} \end{pmatrix} b = \begin{pmatrix} \frac{33}{4} \\ \frac{3}{2} \\ \frac{69}{4} \end{pmatrix}$$
$$A = \begin{pmatrix} \frac{1}{16} & -\frac{1}{8} & \frac{5}{16} \\ \frac{3}{8} & -\frac{1}{4} & -\frac{1}{8} \\ -\frac{3}{16} & -\frac{5}{8} & \frac{1}{16} \end{pmatrix} v = 27$$

Finally, we rewrite equation 4d in terms of x_2 to obtain the final slack form.

$$z = 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \tag{5a}$$
$$x_1 = 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \tag{5b}$$
$$x_2 = 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \tag{5c}$$
$$x_4 = 18 - \frac{x_3}{2} + \frac{x_5}{2} \tag{5d}$$

Now if we set all non-basic variables to 0, we obtain the basic solution 28. Since all the coefficients are negative for the basic variables, we can perform no further work and we are done.

2.1 Algorithm

Before presenting the algorithm, we present the most important step. The PIVOT function will take us from one slack form to another.

PIVOT($\mathbf{N}, \mathbf{B}, \mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{v}, \mathbf{l}, \mathbf{e}$)

$\star$ Compute the coefficients of the equation for new basic variable x_e.

$\hat{b_e} \leftarrow \frac{b_l}{a_{le}}$

for each $j \in \mathbf{N} - \{\mathbf{e}\}$

..... **do** $\hat{a_{ej}} \leftarrow \frac{a_{lj}}{a_{le}}$

$\hat{a_{el}} \leftarrow \frac{1}{a_{le}}$

$\star$ Compute the coefficients of the remaining constraints.

for each $i \in \mathbf{B} - \{\mathbf{l}\}$

..... **do** $\hat{b_i} \leftarrow b_i - a_{ie}\hat{b_e}$

.... **for** each $j \in \mathbf{N} - \{\mathbf{e}\}$

.... **do** $\hat{a_{ij}} \leftarrow a_{ij} - a_{ie}\hat{a_{ej}}$

.... $\hat{a_{il}} \leftarrow -a_{ie}\hat{a_{el}}$

$\star$ Compute the objective function.

$\hat{v} \leftarrow v + c_e\hat{b_e}$

for each $j \in \mathbf{N} - \{\mathbf{e}\}$

..... **do** $\hat{c_j} \leftarrow -c_e\hat{a_{ej}}$

$\hat{c_l} \leftarrow -c_e\hat{a_{el}}$

$\star$ Compute new sets of basic and nonbasic variables.

$\hat{\mathbf{N}} = \mathbf{N} - \{\mathbf{e}\} \cup \{\mathbf{l}\}$

$\hat{\mathbf{B}} = \mathbf{B} - \{\mathbf{l}\} \cup \{\mathbf{u}\}$

return $(\hat{\mathbf{N}}, \hat{\mathbf{B}}, \hat{\mathbf{A}}, \hat{\mathbf{b}}, \hat{\mathbf{c}}, \hat{\mathbf{v}})$

Now that we have defined the PIVOT function, the SIMPLEX algorithm is fairly easy to understand.

SIMPLEX$(\mathbf{A}, \mathbf{b}, \mathbf{c})$

1. $(\mathbf{N}, \mathbf{B}, \mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{v}) \leftarrow$ INITIALIZE-SIMPLEX$(\mathbf{A}, \mathbf{b}, \mathbf{c})$
2. **while** some index $j \in \mathbf{N}$ has $c_j > 0$
3. ..... **do** choose and index $e \in \mathbf{N}$ for which $c_e > 0$
4. .... **for** each index $i \in \mathbf{B}$
5. .... **do if** $a_{ie} > 0$
6. .... **then** $\Delta_i \leftarrow \frac{b_i}{a_{ie}}$
7. .... **else** $\Delta_i \leftarrow \infty$
8. .... choose and index $l \in \mathbf{B}$ that minimizes Δ_i
9. .... **if** $\Delta_l = \infty$
10. .... **then return** ”unbounded”
11. .... **else** $(\mathbf{N}, \mathbf{B}, \mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{v}) \leftarrow$ PIVOT$(\mathbf{N}, \mathbf{B}, \mathbf{A}, \mathbf{b}, \mathbf{c}, \mathbf{v}, \mathbf{l}, \mathbf{e})$
12. **for** $i \leftarrow$ **to** n
13. ..... **do if** $i \in \mathbf{B}$
14. .... **then** $\bar{x}_i \leftarrow b_i$
15. .... **else** $\bar{x}_i \leftarrow 0$
16. **return** $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$

2.2 Lemmas

Below are some of the necessary lemmas to understand why SIMPLEX algorithm produces an optimal result.

Lemma 1. *If SIMPLEX returns a feasible solution, then it is a feasible solution of the linear program. If it returns ”unbounded”, then the linear program is ”unbounded”.*

Lemma 2. *If variables are chosen with lowest index for identifying basic and non-basix swaps (lines 3 and 8 of the SIMPLEX algorithm), then SIMPLEX must terminate.*

Lemma 3. *The set of basic variables determines the slack form uniquely.*

Lemma 4. *If SIMPLEX fails to terminate in at most $\binom{n+m}{m}$ iterations, then it cycles.*