Randomized Algorithms

Advanced Algorithms (CSE 794)
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1 Global MIN-CUT problem

Let $G = (V, E)$ be an undirected graph.

Definition 1. *A cut in a graph G is a partition $(U, V - U)$ of the vertices V.*

Definition 2. *The size of the cut $(U, V - U)$ is the number of edges connecting U and V.*

Global MIN-CUT problem: *Find a cut in an undirected graph of minimal size*

The Global MIN-CUT problem can be solved deterministically using the algorithms for MAX-FLOW as:

- Construct a directed graph $\vec{G} = (V, \vec{E})$ where each edge (u, v) corresponds to two edges (u, v) and (v, u) of $\vec{E}$.
- Assign capacity one to all edges of $\vec{G}$.
- Choose a source vertex s of the the grapph $\vec{G}$.
- Solve the MAX-FLOW problem in $\vec{G}$ sor all sinks $t \in V - s$.

The maximal flow in $\vec{G}$ is the size of the cut (S, T) with minimal capacity where $s \in S$ and $t \in T$. While the deterministic algorithms always produce the optimal solution, the randomized algorithms are often easier to implement and usually run much faster than the deterministic algorithms. Many randomized algorithms often produce the optimal solution and we can give theoretical guarantees for their outcomes. A good randomized algorithm which solves the Global MIN-CUT problem is the *Contraction Algorithm*.

Definition 3. *A contraction in an undirected graph $G = (V, E)$ by an edge (u, v) is a graph $G' = (V', E')$ where $V' = V - \{u, v\} \cup \{x\}$ and x is a new vertex. The set of edges E' is formrd from E by deleting the edges (u, v) and for each vertex w incident to u or v, deleting whichever of (u, w) and (v, w) is in E and adding the new edge (x, w).*

The contraction operation produces a *multigraph* with multiple links between two vertices. The vertices of G' are *supernodes* which represent the contraction operation. After $n - 2$ contractions the multigraph G' consists of two nodes which represent a cut of the graph G and $|E'|$ is equal to the size of the cut. An example of two contractions on a graph with four vertices and five edges is given below.

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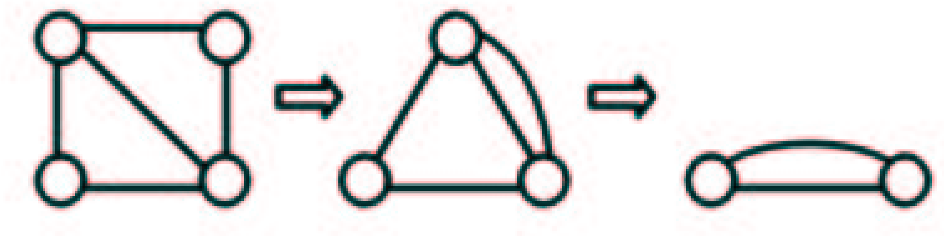

An efficient ranomized algorithm which solves the Global Min-CUT problem is the *Contraction Algorithm*. The main idea of the algorithm is to choose an edge of the multigraph G' at random and perform a contraction operation on G' using this edge.

CONTRACT[$G = (V, E)$**]**
Comment: *for each node* v*, let* $s(v)$ *denote the set contracted to* v*.*
Initialize: $s(v) := v$
for *each* v
    if *G has two nodes* v_1 *and* v_2 ***then***
        return $s(v_1)$ *and* $s(v_2)$
    else *choose an edge* $e = (u, v)$ *of G at random.*
        Let G' *be the contracted multigraph with new node* x_{uv} *replacing* u *and* v*.*
        $S(z_{uv}) := S(u) \cup S(v)$
        Contract[G'**]**
    end if

Theorem 1. *The contraction algorithm returns a global min-cut of G with probability* $\frac{1}{\binom{n}{2}}$.

Proof. Let F be the edges of a global min-cut of size k. Then every vertex of G has degree $\geq k$. The sum of the degrees of all vertices is $\geq nk$. Therefore

$$|V| \geq \frac{kn}{2}$$

The probability that an edge of F is contracted is $\leq \frac{k}{\frac{kn}{2}} = \frac{2}{n}$.

- Consider the multigraph G' which is obtained on the j^{th} iteration by the Contraction Algorihm.
- The graph G' has $n - j$ vertices.
- The probability that an edge from F is contracted on the j^{th} iteration is $\leq \frac{2}{n-j}$.
- Let $\mathscr{E}_j$ be the event that an edge in F is not contracted at iteration j.
- $P(\mathscr{E}_1) \geq 1 - \frac{2}{n} = \frac{n-2}{n}$
- $P(\mathscr{E}_j | \mathscr{E}_1 \cap \mathscr{E}_2 \cap \cdots \cap \mathscr{E}_{j-1}) \geq 1 - \frac{2}{n-j} = \frac{n-j-2}{n-j}$
- $P(\mathscr{E}_1 \cap \mathscr{E}_2 \cap \cdots \cap \mathscr{E}_{n-2}) \geq \frac{n-2}{n}\frac{n-3}{n-1} \cdots \frac{1}{3} = \frac{2}{n(n-1)} = \frac{1}{\binom{n}{2}}$

□

If we run the Contraction Algorithm $\binom{n}{2}$ times the probability that a MIN-CUT is not produced is at most

$$\left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2}} \leq \frac{1}{e}$$

If we run the Contraction Algorithm $\binom{n}{2} \log n$ times the probability is at most

$$\left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2} \log n} \leq \frac{1}{e^{\log n}} = \frac{1}{n}$$

The probability that a Global MIN-CUT is not produced in $O(n^3)$ runs is very small.

2 3-SAT

- $x = \{x_1, \cdots, x_n\}$ boolean variables
- $t_i \in \{x_i, \bar{x}_i\}$
- $C_1, \cdots, C_k$ clauses where $C_i = t_{i_1} \vee t_{i_2} \vee t_{i_3}$

Problem (3-SAT) *Set an assignment to x such that $C_i = true$ for all $i = 1, \cdots, k$.*

Cook-Levin Theorem *3-SAT is NP-complete.*

3-SAT is the first known NP-complete problem (Cook 1970). The 2-SAT problem is easier and it has a polynomial time solution. A related problem is the maximum satisfiability problem.

Max 3-SAT *Find an assignment for x that satisfies the largest number of clauses.*

Let Z_i be the random variable which takes values 0 or 1 defined as

$$Z_i = \begin{cases} 1 & \text{if } c_i \text{ is satisfied} \\ 0 & \text{otherwise} \end{cases}$$

Let Z be the random variable $Z = Z_1 + \cdots + Z_k$. The value of Z is the number of satisfied clauses. Now we calculate the expected value of Z.

Proposition 1. $E(Z) = \dfrac{7k}{8}$

Proof. First we calculate the expected value of Z_i.

$$E(Z_i) = P(C_i \text{ is satisfied}) = 1 - P(C_i \text{ is not satisfied})$$

$$E(Z_i) = 1 - P(t_{i_1} = 0)P(t_{i_2} = 0)P(t_{i_3} = 0) = 1 - \left(\frac{1}{2}\right)^3 = \frac{7}{8}$$

Then

$$E(Z) = E(Z_1) + \cdots + E(Z_k) = \frac{7k}{8}$$

□

Proposition 2. *There exists an assignment which satisfies at least* $\frac{7k}{8}$ *clauses.*

Proof. Let l be the maximal number of clauses satisfied by an assignment of x. Then $E(Z) \leq l$. From Proposition 1: $l \geq \frac{7k}{8}$. □