1.3 Voronoi and Delaunay diagrams

Voronoi diagrams and Delaunay triangulations are important geometric data structures that are built on the notion of 'nearness'. Many differential properties of curves and surfaces are defined on local neighborhoods. Voronoi diagrams and their duals Delaunay triangulations provide a tool to approximate these neighborhoods in the discrete domain. They are defined for a point set in any Euclidean space. We define them in two dimensions and mention the extensions to three dimensions since the curve and surface reconstruction as dealt in this book are concerned with these two Euclidean spaces. Before we go into the definitions we state a non-degeneracy condition for the point set P defining the Voronoi and Delaunay diagrams. This non-degeneracy condition not only makes the definitions less complicated but also makes the algorithms avoid special cases.

Definition 1.2 *A point set* $P \subset \mathbb{R}^k$ *is* non-degenerate *(i) if the affine hull of any* ℓ *points from* P *with* $1 \leq \ell \leq k$ *is homeomorphic to* $\mathbb{R}^{\ell-1}$*, (ii) no* $k+2$ *points are co-spherical.*

1.3.1 Two dimensions

Let P be a set of non-degenerate points in the plane $\mathbb{R}^2$.

Voronoi diagrams. The Voronoi cell V_p for each point $p \in P$ is given as

$$V_p = \{x \in \mathbb{R}^2 \mid d(x, P) = \|x - p\|\}.$$

In words, V_p is the set of all points in the plane that have no other point in P closer to it than p. For any two points p, q the set of points closer to p than q are demarked by the perpendicular bisector of the segment pq. This means the Voronoi cell V_p is the intersection of halfplanes determined by the perpendicular bisectors between p and each other point $q \in P$. Implication of this observation is that each Voronoi cell is a convex polygon since the intersection of convex sets remains convex.

Voronoi cells have *Voronoi faces* of different dimensions. A Voronoi face of dimension k is the intersection of $3-k$ Voronoi cells. This means a k-dimensional Voronoi face for $k \leq 2$ is the set of all points that are equidistant from $3-k$ points in P. A zero dimensional Voronoi face, called *Voronoi vertex* is equidistant from three points in P, whereas an one dimensional Voronoi face, called *Voronoi edge* contains points that are equidistant from two points in P. A Voronoi cell is a two dimensional Voronoi face.

Definition 1.3 *The Voronoi diagram* Vor P *of* P *is the* 2*-complex formed by Voronoi faces.*

Figure 1.9 (a) shows a Voronoi diagram of a point set in the plane where u and v are two Voronoi vertices and uv is a Voronoi edge.

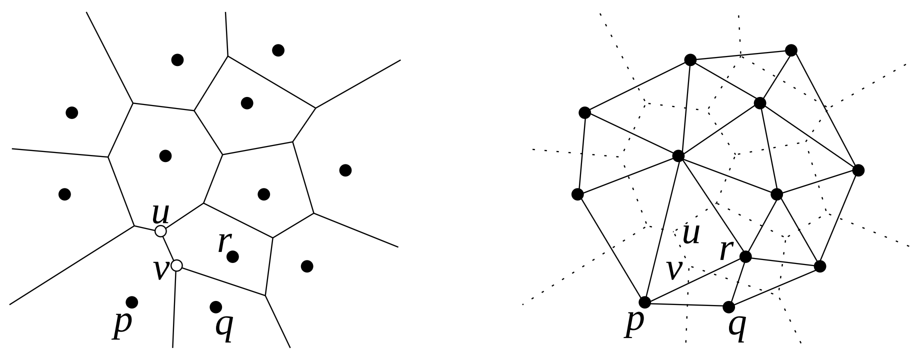

Figure 1.9: (a) The Voronoi diagram, and (b) the Delaunay triangulation of a point set in the plane.

Some of the Voronoi cells may be unbounded with unbounded edges. It is a straightforward consequence of the definition that a Voronoi cell V_p is unbounded if and only if p is on the boundary of the convex hull of P. In Figure 1.9 (a) V_p and V_q are unbounded and p and q are on the convex hull boundary.

Delaunay triangulations. There is a *dual* structure to the Voronoi diagram Vor P, called the *Delaunay triangulation* denoted Del P. Formally, we define Del P as a simplicial complex where

$$\operatorname{Del} P = \{\sigma = \operatorname{Conv} T \mid \bigcap_{p \in T} V_p \neq \emptyset\}.$$

In words, $k+1$ points in P form a Delaunay k-simplex in Del P if their Voronoi cells have nonempty intersection. We know that $k+1$ Voronoi cells meet in a $(2-k)$-dimensional Voronoi face. So, each k-simplex in Del P is dual to a $(2-k)$-dimensional Voronoi face. Thus, each Delaunay triangle pqr in Del P is dual to a Voronoi vertex where V_p, V_q and V_r meet; each Delaunay edge pq is dual to a Voronoi edge shared by Voronoi cells V_p and V_q, and each vertex p is dual to its corresponding Voronoi cell V_p. In Figure 1.9 (b), the Delaunay triangle pqr is dual to the Voronoi vertex v and the Delaunay edge pr is dual to the Voronoi edge uv. In general, when μ is a dual Voronoi face of a Delaunay simplex σ we say $\mu = \text{dual}\,\sigma$ and conversely $\sigma = \text{dual}\,\mu$.

A *circumscribing ball* of a simplex σ is a ball whose boundary contains the vertices of the simplex. The smallest circumscribing ball of σ is called its *diametric* ball. For triangles in the plane there is only one circumscribing ball, namely the diametric one. However, for edges there are infinitely many circumscribing balls among which the diametric one is unique, namely the one with the edge as diameter.

A dual Voronoi vertex of a Delaunay triangle is equidistant from its three vertices. This means that the center of the circumscribing ball of a Delaunay triangle is the dual Voronoi vertex. It implies that no point from P can lie in the interior of the circumscribing ball of a Delaunay triangle. These balls are called *Delaunay*. A ball is *empty* if its interior does not contain any point from P. Clearly, the Delaunay balls are *empty*. The converse also holds.

Property 1.1 (Triangle emptiness.) *A triangle is in the Delaunay triangulation if and only if its circumscribing ball is empty.*

The Triangle Emptiness Property of Delaunay triangles also implies a similar emptiness for Delaunay edges. Clearly, each Delaunay edge has an empty circumscribing ball passing through its endpoints. It turns out that the converse is also true, that is, any edge pq with an empty cisrcumscribing ball must also be in the Delaunay triangulation. To see this, grow the empty ball of pq always keeping p, q on its boundary. If it never meets any other point from P, the edge pq is on the convex hull boundary of P and is in the Delaunay triangulation since V_p and V_q has to share an edge extending to infinity. Otherwise, when it meets a third point, say r from P, we have an empty circumscribing ball passing through p, q, r. By the Triangle Emptiness Property pqr must be in the Delaunay triangulation and hence the edge pq.

Property 1.2 (Edge emptiness.) *An edge is in the Delaunay triangulation if and only if the edge has an empty circumscribing ball.*

The Delaunay triangulation form a planar graph since no two Delaunay edges intersect in their interiors. It follows from the property of planar graphs that the number of Delaunay edges is at most $3n - 6$ for a set of n points. The number of Delaunay triangles is at most $2n - 4$. This means that the dual Voronoi diagram also has at most $3n - 6$ Voronoi edges and $2n - 4$ Voronoi vertices. The Voronoi diagram and the Delaunay triangulation of a set of n points in the plane can be computed in $O(n \log n)$ time and $O(n)$ space.

Restricted Voronoi diagrams. When the input point set P is a sample of a curve or a surface Σ, we have structures imposed by the Voronoi diagram $\mathrm{Vor}\, P$ on Σ. It turns out that this diagram plays an important role in reconstructing Σ from P. Formally, a restricted Voronoi cell $V_p|_\Sigma$ is defined as the intersection of the Voronoi cell V_p in $\mathrm{Vor}\, P$ with Σ, i.e.,

$$V_p|_\Sigma = V_p \cap \Sigma \text{ where } p \in P.$$

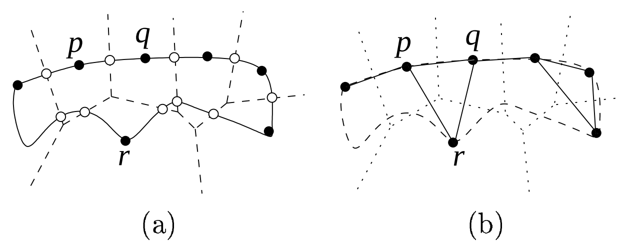

Figure 1.10: (a) Restricted Voronoi diagram for a point set on a curve, (b) the corresponding restricted Delaunay triangulation.

Similar to the Voronoi diagram, we can define *restricted Voronoi faces* as the intersection of the restricted Voronoi cells. They can also be viewed as the intersection of Voronoi faces with Σ. In Figure 1.10 (a) the white circles represent restricted Voronoi faces of dimension zero. The curve segments between them are restricted Voronoi faces of dimension one which are restricted Voronoi cells in this case. Notice that the restricted Voronoi cell $V_p|_\Sigma$ consists of two curve segments whereas $V_r|_\Sigma$ consists of a single curve segment. The restricted Voronoi diagram $\mathrm{Vor}\, P|_\Sigma$ of P with respect to Σ is the collection of all restricted Voronoi faces.

Restricted Delaunay triangulations. As with Voronoi diagrams we can define a simplicial complex $\mathrm{Del}\, P|_\Sigma$ dual to a restricted Voronoi diagram $\mathrm{Vor}\, P|_\Sigma$. A k-simplex in this dual complex is defined with $k+1$ vertices, R, where

$$\bigcap V_p|_\Sigma \neq \phi, \text{ for } p \in R.$$

The simplicial complex $\mathrm{Del}\, P|_\Sigma$ is called the *restricted Delaunay triangulation* of P with respect to Σ. Figure 1.10 (b) shows the restricted Delaunay triangulation for the restricted Voronoi diagram in (a). The vertex p is connected to q and r in the restricted Delaunay triangulation since $V_p|_\Sigma$ meets both $V_q|_\Sigma$ and $V_r|_\Sigma$. However the triangle pqr is not in the triangulation since $V_p|_\Sigma$, $V_q|_\Sigma$ and $V_r|_\Sigma$ do not meet at a point.

1.3.2 Three dimensions

We chose the plane to explain the concepts of the Voronoi diagrams and the Delaunay triangulations in the previous discussions. However, these concepts extend to arbitrary dimensions. We will mention these extensions for three dimensions which will be important for later expositions.

Voronoi cells of a point set P in $\mathbb{R}^3$ are three dimensional convex polytopes some of which are unbounded. There are four types of Voronoi faces; Voronoi vertex, Voronoi edge, Voronoi facet and Voronoi cell in increasing order of dimension starting with zero and ending with three. Four Voronoi cells meet at a Voronoi vertex which is equidistant from four points in P. Three Voronoi cells meet at a Voronoi edge and two Voronoi cells meet at a Voronoi facet.

The Delaunay triangulation of P contains four types of simplices dual to each of the four types of Voronoi faces. The vertices are dual to the Voronoi cells, the Delaunay edges are dual to the Voronoi facets, the Delaunay triangles are dual to the Voronoi edges, and the Delaunay tetrahedra are dual to the Voronoi vertices. The circumscribing ball of each tetrahedron is empty. Conversely, any tetrahedron with empty circumscribing ball is in the Delaunay triangulation. Further, each Delaunay triangle and edge has an empty circumscribing ball. Conversely, an edge or a triangle belongs to the Delaunay triangulation if there exists an empty ball circumscribing it.

The number of edges, triangles and tetrahedra in the Delaunay triangulation of a set of n points in three dimensions can be $O(n^2)$ in the worst case. By duality the Voronoi diagram can also have $O(n^2)$ space complexity. Both of the diagrams can be computed in $O(n^2)$ time and space.

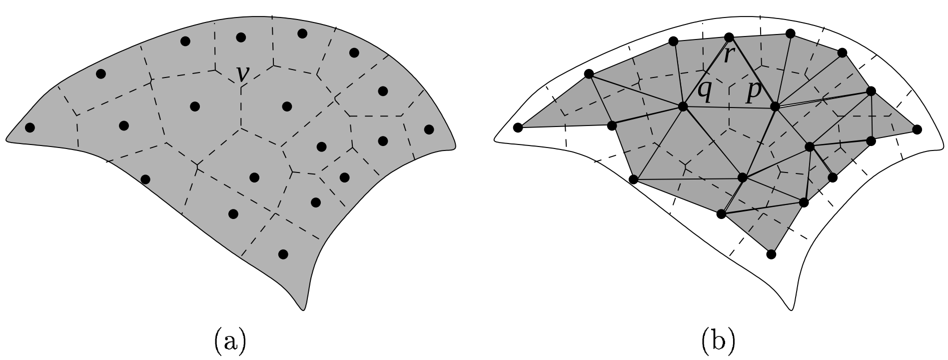

Figure 1.11: (a) The restricted Voronoi diagram, and (b) the restricted Delaunay triangulation for a sample on a surface.

We can define the restricted Voronoi diagram and its dual restricted Delaunay triangulation for a point sample on a surface in $\mathbb{R}^3$ in the same way as we did for a curve in $\mathbb{R}^2$. Figure 1.11 shows the restricted Voronoi diagram and its dual restricted Delaunay triangulation for a set of points on a surface. The triangle pqr is in the restricted Delaunay triangulation since $V_p|_\Sigma$, $V_q|_\Sigma$ and $V_r|_\Sigma$ meet at a common point v.