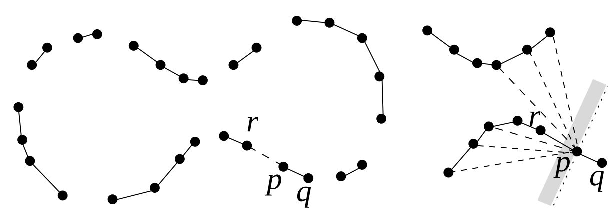
2.3 NN-crust

The next algorithm for curve reconstruction is based on the concept of nearest neighbors. A point $p \in P$ is a nearest neighbor of $q \in P$ if there is no other point $s \in P \setminus \{p, q\}$ with $\|q - s\| < \|q - p\|$. Notice that p being a nearest neighbor of q does not necessarily mean that q is a nearest neighbor of p.

We first observe that edges that connect nearest neighbors in P must be correct edges if P is sufficiently dense. But, all correct edges do not connect nearest neighbors. Figure 2.7 shows all edges that connect nearest neighbors. The missing correct edges in this example connect points that are not nearest neighbors. However, these correct edges connect points that are not very far from being nearest neighbors. We capture them in NN-CRUST using the notion of *half neighbors*.



(a) (b)

Figure 2.7: (a) Only nearest neighbor edges may not reconstruct a curve, (b) half neighbor edges fill up the gaps such as pr.

2.3.1 Algorithm

Let pq be an edge connecting p to its nearest neighbor q and $\vec{pq}$ be the vector from p to q. Consider the closed halfplane H bounded by the line passing through p with $\vec{pq}$ as outward normal. Clearly, $q \notin H$. The nearest neighbor to p in the set $H \cap P$ is called its *half neighbor*. In Figure 2.7(b), r is the half neighbor of p. It can be shown that two correct edges incident to a sample point connect it to its nearest and half neighbors.

The above discussion immediately suggests an algorithm for curve reconstruction. But, we need efficient algorithms to compute nearest neighbor and half neighbor for each sample point. The Delaunay triangulation Del P turns out to be useful for this computation as all correct edges are Delaunay if P is sufficiently dense. Small Edge Lemma (2.3) implies that, for each sample point p, it is sufficient to check only the Delaunay edges to determine correct edges. We check all edges incident to p in $\mathrm{Del}\,P$ and determine the shortest edge connecting it to its nearest neighbor, say q. Next, we check all other edges incident to p which make at least $\frac{\pi}{2}$ angle with pq at p and choose the shortest among them. This second edge connects p to its half neighbor. The entire computation can be done in time proportional to the number of edges incident to p. Since the sum of the number of incident edges over all vertices in the Delaunay triangulation is $O(n)$ where $|P| = n$, correct edge computation takes only $O(n)$ time once $\mathrm{Del}\,P$ is computed. The Delaunay triangulation of a set of n points in the plane can be computed in time $O(n \log n)$ which implies that NN-crust takes $O(n \log n)$ time.

NN-CRUST (P)
compute $\mathrm{Del}\,P$;
$E = \phi$;
`for` each $p \in P$ `do`
   compute the shortest edge pq in $\mathrm{Del}\,P$;
   compute the shortest edge ps so that $\angle pqs \geq \frac{\pi}{2}$;
   $E = E \cup \{pq, ps\}$;
`endfor`
`return` E.

2.3.2 Correctness

As we discussed before, NN-CRUST computes edges connecting each sample point to its nearest and half neighbors. The correctness of NN-CRUST follows from the proofs that these edges are correct.

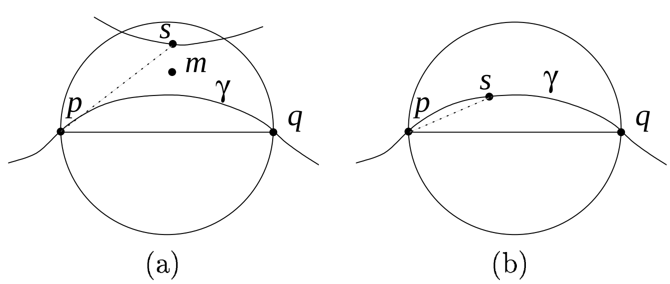


Figure 2.8: Diametric ball of pq intersects Σ in (a) two components, (b) single component.

Lemma 2.8 (Neighbor.) *Let $p \in P$ be any sample point and q be its nearest neighbor. The edge pq is correct for $\varepsilon < \frac{1}{3}$.*

PROOF. Consider the ball B with pq as diameter. If B does not intersect Σ in a 1-ball, it contains a medial axis point by Feature Ball Lemma (1.1). See Figure 2.8(a). This means $\|p - q\| > f(p)$. A correct edge ps satisfies $\|p - s\| \leq \frac{2\varepsilon}{1-\varepsilon} f(p)$ by Small Edge Lemma (2.3). Thus, for $\varepsilon < \frac{1}{3}$ we have $\|p - s\| < \|p - q\|$, a contradiction to the fact that q is the nearest neighbor to p.

So, B intersects Σ in a 1-ball, namely $\gamma = \gamma(p, q)$ as shown in Figure 2.8(b). If pq is not correct, γ contains a sample point, say s, between p and q inside B. Again, we reach a contradiction as $\|p - s\| < \|p - q\|$. □

Next we show that edges connecting a sample point to its half neighbors are also correct.

Lemma 2.9 (Half Neighbor.) *An edge pq where q is a half neighbor of p is correct when $\varepsilon < \frac{1}{3}$.*

PROOF. Let r be the nearest neighbor of p. According to the definition $\vec{pq}$ makes at least $\frac{\pi}{2}$ angle with $\vec{pr}$.

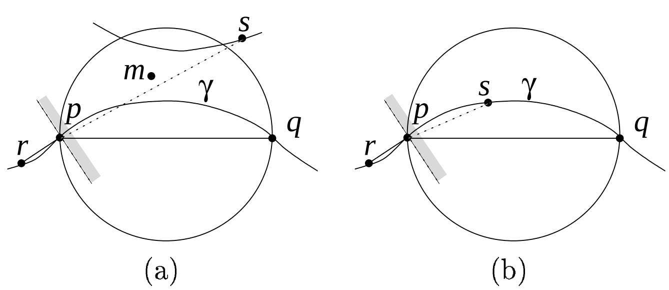


Figure 2.9: (a) Diametric ball of pq intersects Σ in (a) more than one component, (b) a single component.

If pq is not correct, consider the correct edge ps incident to p other than pr. By Edge Angle Lemma (2.5) $\vec{ps}$ also makes at least $\frac{\pi}{2}$ angle with $\vec{pr}$ for $\varepsilon < 1/3$. We show that s is closer to p than q. This contradicts that q is the half neighbor of p since both $\vec{ps}$ and $\vec{pq}$ make an angle at least $\frac{\pi}{2}$ with $\vec{pr}$.

Consider the ball B with pq as a diameter. If B does not intersect Σ in a 1-ball (Figure 2.9(a)), it would contain a medial axis point, and thus

$\|p-q\| \geq f(p)$. On the other hand $\|p-s\| \leq \frac{2\varepsilon}{1-\varepsilon} f(p)$ by Small Edge Lemma (2.3). We get $\|p-s\| < \|p-q\|$ for $\varepsilon < \frac{1}{3}$ as required for contradiction. Next, assume that B intersects Σ in a 1-ball, namely in $\gamma(p,q)$, as in Figure 2.9(b). Since pq is not a correct edge, s must be on this curve segment. It implies $\|p-s\| < \|p-q\|$ as required for contradiction. □

Theorem 2.2 NN-CRUST *computes all and only correct edges when* $\varepsilon < \frac{1}{3}$.

PROOF. By Small Edge Lemma (2.3) all correct edges are Delaunay. Step 4 and 5 assure that all edges joining samples points to their nearest and half neighbors are computed as output. These edges are correct by Neighbor Lemma (2.8) and Half Neighbor Lemma (2.9) when $\varepsilon < \frac{1}{3}$. Also, there is no other correct edges since each sample point can only be incident to exactly two correct edges.