## 4.4 Notes and exercises

The problem of reconstructing surfaces from samples dates back to early 1980s. First, the problem appeared in the form of contour surface reconstruction in medical imaging. A set of cross sections obtained via CAT scan or MRI need to be joined with a surface in this application. The points on the boundary of the cross sections are already joined by a polygonal curve. The problem is to connect these curves in consecutive cross sections. A dynamic programming based solution for two such consecutive curves was first proposed by Fuchs, Kedem and Uselton [FKU77]. A result by Gitlin, O'Rourke and Subramanian [GOS96] shows that, in general, two polygonal curves cannot be joined by non-self intersecting surface with only those vertices; even deciding its possibility is NP-hard. Several solutions with the addition of Steiner points have been proposed to overcome the problem, see Meyers, Skinner and Sloan [MSS92]. A Delaunay based solution for the problem was proposed by Boissonnat [Boi84] which is the first Delaunay based algorithm proposed for a surface reconstruction problem. Later the Delaunay based method was refined by Boissonnat and Geiger [BG93] and Cheng and Dey [CD99].

The most general version of the surface reconstruction where no input information other than the point co-ordinates is used became popular to handle the data from range and laser scanners. In the context of computer graphics and vision, this problem has been investigated intensely in the past decade with emphasis on the practical performances. The early work by Hoppe et al. [HDDMS92], Curless and Levoy [CL96] and the recent work by Alexa et al. [ABCFLS01] and Ohtake et al. [OBATS03] are few such examples. The $\alpha$-shape by Edelsbrunner and Mücke [EM94] is the first popular Delaunay based surface reconstruction method. Depending on an input parameter $\alpha$, Delaunay simplices are filtered based on their circumscribing Delaunay ball sizes. The main drawback of this method is that it is not suitable for nonuniform samples. Also, with the uniform samples, the user is burdened with the choice of an appropriate $\alpha$.

The first provable algorithm for surface reconstruction was devised by Amenta and Bern [AB99]. They generalized the CRUST algorithm for curve reconstruction to the surface reconstruction problem. The idea of poles and approximating the normals with the pole vector was a significant breakthrough. The crust triangles (exercise 2) enjoy some nice properties that help the reconstruction. The COCONE algorithm as described here is a successor of CRUST. Devised by Amenta, Choi, Dey and Leekha [ACDL02], this algorithm simplified the CRUST algorithm and its proof of correctness. CO-

CONE eliminated one of the two Voronoi diagram computations of CRUST and also a normal filtering step. The homeomorphism between the reconstructed surface and the original sampled surface was first established in [ACDL02]. Boissonnat and Cazals [BC00] devised another algorithm for surface reconstruction using the natural neighbor co-ordinates and proved its correctness using the framework of CRUST. Since the Deluanay triangulations of $n$ points in three dimensions take $O(n^2)$ time and space in the worst-case, the complexity of all these algorithms is $O(n^2)$. Funke and Ramos [FR02] showed how the COCONE algorithm can be adapted to run in $O(n \log n)$ time. Unfortunately, the modified algorithm is not very practical.

## Exercises

1. We know that Voronoi vertices for a dense sample from a curve in the plane lie near the medial axis. The same is not true for surfaces in three dimensions. Show an example where a Voronoi vertex for an arbitrarily dense sample lies arbitrarily close to the surface.

2. Let $P$ be a sample from a smooth surface $\Sigma$ and $V$ be the set of poles in $\mathrm{Vor}\, P$. Consider the following generalization of the CRUST. A triangle in the $\mathrm{Del}\,(P \cup V)$ is a crust triangle if all of its vertices come from $P$. Show the following when $P$ is an $\varepsilon$-sample for a sufficiently small $\varepsilon$.

   (i) All restricted Delaunay triangles in $\mathrm{Del}\,(P \cup V)|_\Sigma$ are crust triangles.

   (ii) All crust triangles have circumradius $\tilde{O}(\varepsilon) f(p)$ where $p$ is a vertex of the triangle.

3. Let $t$ be a triangle in $\mathrm{Del}\, P$ where $B = B_{v,r}$ and $B' = B'_{v',r'}$ are two Delaunay balls circumscribing $t$. Let $x$ be any point on the circle where the boundaries of $B$ and $B'$ intersect. Show that, if $\angle vxv' > \frac{\pi}{2}$, the triangle normal of $t$ makes an angle of $\tilde{O}(\varepsilon)$ with the normals to $\Sigma$ at its vertices when $P$ is an $\varepsilon$-sample of $\Sigma$ for sufficiently small $\varepsilon$.

4. Call $P$ a locally $(\varepsilon, \delta)$-uniform sample of a smooth surface $\Sigma$ if $P$ is an $\varepsilon$-sample of $\Sigma$ and further each sample point $p \in P$ is at least $\frac{\varepsilon}{\delta} f(p)$ distance away from all other points in $P$ where $\delta > 0$ is a constant. Show that each triangle in the surface output by COCONE for such a sample has a bounded aspect ratio. Also prove that each vertex has no more than a constant number (determined by $\varepsilon$ and $\delta$) of triangles on the surface.

5. Let $t$ be a cocone triangle. Prove that any point $x \in t$ is $\tilde{O}(\varepsilon^2)f(\tilde{x})$ away from its closest point $\tilde{x}$ in $\Sigma$.

6. Prove that the surface $E$ computed by COCONE is isotopic to $\Sigma$ when $\varepsilon$ is sufficiently small.