2.4 Notes

In its simplest form the curve reconstruction problem appears in applications such as pattern recognition, image boundary detection, and cluster analysis. In the 1980s several geometric graphs connecting a set of points in the plane were discovered which reveal a pattern among the points. The influence graph of Toussaint [AH85], the β-skeleton of Kirkpatrick and Radke [KR85], the α-shapes of Edelsbrunner, Kirkpatrick, Seidel [EKS83] are such graphs.

A set of points from a curve Σ is called a *δ-uniform sample* if each point $x \in \Sigma$ has a sample point within a fixed distance δ. Several algorithms were devised to reconstruct curves from δ-uniform samples with δ being sufficiently small. Attali proposed a Delaunay based reconstruction for such samples [Att97] (also see exercise 4). Figueiredo and Gomes [FG95] showed that Euclidean minimum spanning tree (EMST) can reconstruct curves with boundaries from sufficiently dense uniform sample. Bernardini and Bajaj [BB97] proved that the α-shapes of Edelsbrunner et al. reconstruct curves from δ-uniform samples with guarantees if δ is sufficiently small.

The first breakthrough in reconstructing curves from non-uniform samples was made by Amenta, Bern and Eppstein [ABE98]. The presented CRUST algorithm is taken from this paper with some modifications in the proofs. Following the development of CRUST, Dey and Kumar devised the NN-CRUST algorithm [DK99]. The presented NN-CRUST algorithm is taken from this paper again with some modifications in the proofs. This algorithm also can reconstruct curves in three and higher dimensions, albeit with appropriate modifications of the proofs (exercise 5).

The CRUST and NN-CRUST assume that the sample is derived from a smooth curve without boundaries. The questions of reconstructing non-smooth curves and curves with boundaries have also been studied.

Giesen [Gie00] showed that a fairly large class of non-smooth curves can be reconstructed by Traveling Salesman Path (or Tour). A curve Σ is called *benign* if the left tangent and the right tangent exist at each point and make an angle less than π. Giesen proved that, a benign curve Σ can be reconstructed from a sufficiently dense uniform sample by the Traveling Salesman Path (or Tour) in case Σ has a boundary (or no boundary). The uniform sampling condition was later removed by Althaus and Mehlhorn [AM02], who also gave a polynomial time algorithm to compute the Traveling Salesman Path (or Tour) in the special case of curve reconstruction. The Traveling Salesman approach cannot handle curves with multiple components. Also, the sample points representing the boundaries need to be known a priori to choose between path or tour.

Dey, Mehlhorn and Ramos [DMR00] presented an algorithm named CONSERVATIVE CRUST that provably reconstructs smooth curves with boundaries. Any algorithm for handling curves with boundaries faces a dilemma when an input point set samples a curve without boundary densely and simultaneously samples densely another curve with boundary. This dilemma is resolved in CONSERVATIVE CRUST by a justification on the output. For any input point set P, the graph output by the algorithm is guaranteed to be the reconstruction of a smooth curve possibly with boundary for which P is a dense sample. The main idea of the algorithm is that an edge pq is output only if there is a large enough ball centering the midpoint of pq which is empty of all Voronoi vertices in $\mathrm{Vor}\, P$. The rationale behind this choice is that these edges are small enough with respect to local feature size of the sampled curve since the Voronoi vertices approximate the medial axis. With a sampling condition tailored to handle non-smooth curves, Funke and Ramos [FR01] and Dey and Wenger [DW02] proposed algorithms to reconstruct non-smooth curves. The algorithm of Funke and Ramos can handle boundaries as well.

2.5 Exercises

1. Give an example of a point set P such that P is an 1-sample of two curves for which the correct reconstructions are different.

2. Given a $\frac{1}{4}$-sample P of a smooth curve, show that all correct edges are Gabriel in $\mathrm{Del}\,(P \cup V)$ where V is the set of Voronoi vertices in $\mathrm{Vor}\, P$.

3. Let P be an ε-sample of a smooth curve without boundary. Let η_{pq} be the sum of the angles opposite to pq in the two (or one if pq is a convex hull edge) triangles incident to pq in $\mathrm{Del}\, P$. Prove that there is an ε for which pq is correct if and only if $\eta_{pq} < \pi$.

4. Show that the NN-CRUST algorithm can reconstruct curves in three dimensions from sufficiently dense samples.

5. Correct Edge Lemma (2.6) is proved for $\varepsilon < \frac{1}{5}$. Show that it also holds for $\varepsilon \leq \frac{1}{5}$. Similarly show that Neighbor Lemma (2.8) and Half Neighbor Lemma (2.9) hold for $\varepsilon \leq 1/3$.

6. Gold and Snoeyink [GS01] showed that the CRUST algorithm can be modified to guarantee a reconstruction with $\varepsilon < 0.42$. Althaus [Alt01] showed that the NN-CRUST algorithm can be proved to reconstruct

curves from ε-samples for $\varepsilon < 0.5$. Can this bound on ε be improved? What is the largest value of ε for which curves can be reconstructed from ε-samples?

7. Let $v \in V_p$ be a Voronoi vertex in the Voronoi diagram $\text{Vor}\, P$ of an ε-sample P of a smooth curve Σ. Show that there exists a point m in the medial axis of Σ so that $\|m - v\| \leq c\varepsilon f(p)$ for ε sufficiently small and an appropriate constant $c > 0$.