3.2.2 Voronoi faces

Next we consider in turn the Voronoi edges, Voronoi facets and Voronoi cells and show that they indeed satisfy the condition for the closed ball property if ε satisfies Condition A. Let

$$\begin{aligned} \alpha(\varepsilon) &= \frac{\varepsilon}{1-3\varepsilon} \\ \beta(\varepsilon) &= \arcsin\frac{\varepsilon}{1-\varepsilon} + \arcsin\left(\frac{2}{\sqrt{3}}\sin\left(2\arcsin\frac{\varepsilon}{1-\varepsilon}\right)\right) \\ \textit{Condition A.}\quad & \cos\left(\alpha(\varepsilon)+\beta(\varepsilon)\right) > \frac{\varepsilon}{1-\varepsilon}. \end{aligned}$$

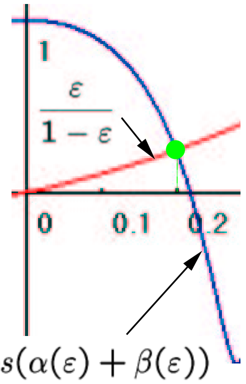


Figure 3.9: The graphs of the two functions on the left and right of the inequality in Condition A.

Figure 3.9 shows that Condition A holds for ε a little less than 0.2. So, for example, $\varepsilon < 0.18$ is a safe choice.

Lemma 3.8 (Voronoi Edge.) *A Voronoi edge intersects Σ transversally in a single point if Condition* A *holds.*

Proof. Suppose, for the sake of contradiction, there is a Voronoi edge e in a Voronoi cell V_p intersecting Σ at two points x and y, or at a single point tangentially, see Figure 3.10. The dual Delaunay triangle, say pqr, is a restricted Delaunay triangle. By Corollary 3.1, its circumradius is no more than $\frac{\varepsilon}{1-\varepsilon}f(p)$. By Triangle Normal Lemma (3.5) $\angle_a(\mathbf{n}_{pqr},\mathbf{n}_p) \le \beta(\varepsilon)$ if

$$\frac{1}{\sqrt{2}} > \frac{\varepsilon}{1-\varepsilon}. \tag{3.1}$$

The Normal Variation Lemma (3.3) puts an upper bound of $\beta(\varepsilon)$ on the angle between the normals at p and x as $\|x-p\| \leq \varepsilon f(x)$. Let ξ denote the angle between $\mathbf{n}_x$ and the Voronoi edge e. We have

$$\begin{aligned} \xi &\leq \angle_a(\mathbf{n}_x, \mathbf{n}_p) + \angle_a(\mathbf{n}_p, \mathbf{n}_{pqr}) \\ &\leq \alpha(\varepsilon) + \beta(\varepsilon). \end{aligned}$$

If e intersects Σ tangentially at x, we have $\xi = \frac{\pi}{2}$ requiring $\alpha(\varepsilon) + \beta(\varepsilon) \geq \frac{\pi}{2}$. This contradicts Condition A as it requires $\varepsilon < 0.2$ and hence $\alpha(\varepsilon) + \beta(\varepsilon) < \frac{\pi}{2}$. Therefore, assume that e intersects Σ at two points x and y.

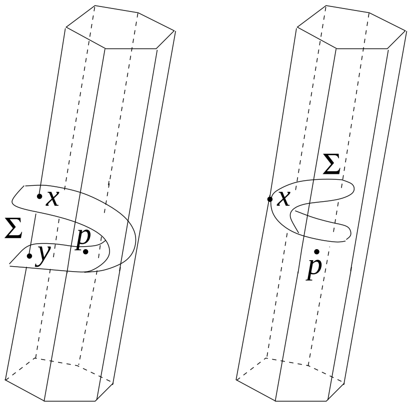


Figure 3.10: Illustration for Voronoi Edge Lemma. Voronoi edge intersects the surface at two points (left) and tangentially in a single point (right).

By Short Distance Lemma (3.6) $\|x-y\| \leq \frac{2\varepsilon}{1-\varepsilon} f(x)$ and by Long Distance Lemma (3.7) $\|x-y\| \geq 2f(x)\cos\xi$. A contradiction is reached when $2\cos\xi > \frac{2\varepsilon}{1-\varepsilon}$, or

$$\cos(\alpha(\varepsilon) + \beta(\varepsilon)) > \frac{\varepsilon}{1-\varepsilon}. \tag{3.2}$$

Condition 3.1 requires that $\varepsilon < \frac{1}{1+\sqrt{2}}$ which is satisfied by Condition A. Therefore, both conditions 3.1 and 3.2 are satisfied by Condition A. □

Lemma 3.9 (Voronoi Facet.) *A Voronoi facet F intersects Σ transversally in a 1-ball if Condition A is satisfied.*

Proof. The intersection of F with Σ may contradict the assertion in the lemma if (i) Σ touches F tangentially at a point, (ii) Σ intersects F in a 1-sphere, that is, a cycle, and (iii) Σ intersects F in more than one component.

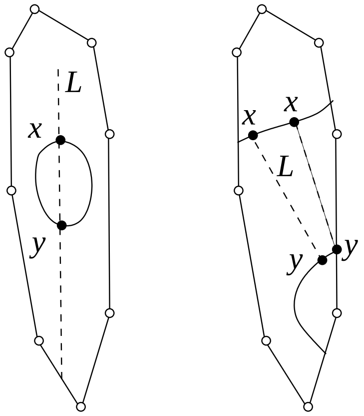
The dual Delaunay edge, say pq, of F is in the restricted Delaunay triangulation. Let $\mathbf{n}_F$ denote the normal to F and its direction is same as that of pq up to orientation. By Edge Normal Lemma (3.4)

$$\angle_a(\mathbf{n}_p, \mathbf{n}_F) \geq \frac{\pi}{2} - \arcsin\frac{\varepsilon}{1-\varepsilon}.$$

If Σ meets F tangentially at a point x, we have $\angle_a(\mathbf{n}_x, \mathbf{n}_F) = 0$ and by the Normal Variation Lemma (3.3) $\angle \mathbf{n}_p, \mathbf{n}_x \leq \frac{\varepsilon}{1-3\varepsilon}$. This means we have

$$\frac{\pi}{2} - \arcsin\frac{\varepsilon}{1-\varepsilon} \leq \angle_a(\mathbf{n}_p, \mathbf{n}_F) \leq \frac{\varepsilon}{1-3\varepsilon} = \alpha(\varepsilon)$$

contradicting the upper bound for ε given by Condition A. If Σ meets F in a cycle, let x be any point on it and L be the line on F intersecting the cycle at x orthogonally, see Figure 3.11(a). This line must meet the cycle in another point, say y. The angle between L and $\mathbf{n}_x$ satisfies $\angle_a(L, \mathbf{n}_x) \leq \angle_a(L', \mathbf{n}_x)$ for any other line L' on F passing through x. Choose L' that minimizes the angle with $\mathbf{n}_p$. Edge Normal Lemma (3.4) implies $\angle_a(L', \mathbf{n}_p) \leq \arcsin\frac{\varepsilon}{1-\varepsilon}$. These facts with the Normal Variation Lemma (3.3) gives



(a) (b)

Figure 3.11: A Voronoi facet intersecting Σ (a) in a cycle, (b) in two segments.

$$\angle_a(L', \mathbf{n}_x) \leq \angle_a(L', \mathbf{n}_p) + \angle(\mathbf{n}_p, \mathbf{n}_x) \leq \arcsin\frac{\varepsilon}{1-\varepsilon} + \alpha(\varepsilon). \qquad (3.3)$$

The right hand side of the inequality 3.3 is less than the upper bound for ξ in the proof of Voronoi Edge Lemma (3.8) and thus we reach a contradiction between distances implied by the Short and Long Distance Lemma (3.7) when Condition A holds.

In the case Σ meets F in two or more components as in Figure 3.11(b), consider any point x in one of the components. Let y be the closest point to x on any other component, say C. If the line L joining x and y meets C orthogonally at y we have the situation as in the previous case with only x and y interchanged. In the other case, y lies on the boundary of C on a Voronoi edge. The angle between L and $\mathbf{n}_y$ is less than the angle between the Voronoi edge and $\mathbf{n}_y$ which is no more than $\alpha(\varepsilon)+\beta(\varepsilon)$ as proved in the Voronoi Edge Lemma (3.8). We reach a contradiction again between two distances using the same argument. $\square$

Lemma 3.10 (Voronoi Cell.) *A Voronoi cell V_p intersects Σ in a 2-ball if Condition A holds.*

PROOF. We have $W = V_p \cap \Sigma$ contained in a ball B of radius $\frac{\varepsilon}{1-\varepsilon} f(p)$ by Short Distance Lemma (3.6). If W is a manifold without boundary, B contains a medial axis point by Feature Ball Lemma (1.1). We reach a contradiction if $\varepsilon < \frac{1}{2}$ which is satisfied by Condition A. So, assume that W is a manifold with boundary. It may not be a 2-ball only if it is non-orientable, or it has a handle, or has more than one boundary cycles. If W were non-orientable, so would be Σ, which is impossible. In case W has a handle, $B \cap \Sigma$ is not a 2-ball. By Feature Ball Lemma (1.1), it contains a medial axis point reaching a contradiction for $\varepsilon < \frac{1}{2}$ which is satisfied by Condition A.

The only possibility left is that W has more than one boundary cycles. Let L be the line the normal at p. Consider a plane that contains L and intersects at least two boundary cycles. Such a plane exists since otherwise L must intersect W at a point other than p and we reach a contradiction between two distance lemmas. The plane intersects V_p in a convex polygon and W in at least two curves. We can argue as in the proof of Voronoi Facet Lemma (3.9) to reach a contradiction between two distance lemmas. $\square$

Theorem 3.2 (Topological Ball.) *For $\varepsilon < 0.18$, $\mathrm{Vor}\, P$ satisfies the topological ball property and hence $\mathrm{Del}\, P|_\Sigma$ is homeomorphic to Σ.*