3.1.2 Normal variation

The directions of the normals at nearby points on Σ cannot vary too abruptly. In other words, the surface looks flat locally. This fact is used later in many proofs.

Lemma 3.3 (Normal Variation.) *Let x and y be any two points with $\|x-y\| \leq \rho f(x)$ for $\rho < \frac{1}{3}$. Then $\angle(\mathbf{n}_x, \mathbf{n}_y) \leq \frac{\rho}{1-3\rho}$.*

PROOF. Let $\ell(t)$ denote any point on the segment xy parameterized by its distance t from x. Let $x(t)$ be the nearest point on Σ from $\ell(t)$. The rate of change of normal $\mathbf{n}_{x(t)}$ at $x(t)$ is $\mathbf{n}'_t = \frac{d\mathbf{n}_{x(t)}}{dt}$ as t changes. The total variation in normals between x and y is

$$\angle(\mathbf{n}_x, \mathbf{n}_y) \leq \int_{xy} |\mathbf{n}'_t| dt \leq \|x-y\| \max_t |\mathbf{n}'_t|.$$

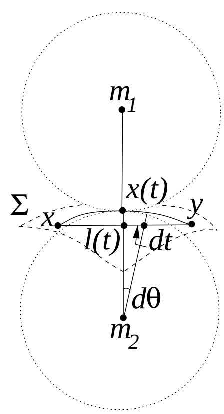

Figure 3.4: Illustration for Normal Variation Lemma.

The surface Σ is squeezed in between two medial balls that are tangent to Σ at $x(t)$. The radii of the two balls cannot be larger than the radius of curvature of Σ at $x(t)$. This means Σ cannot turn faster than the larger of the two medial balls at $x(t)$, refer to Figure 3.4. We have

$$dt \geq (f(x(t)) - \|x(t) - \ell(t)\|) \sin d\theta.$$

As $\sin d\theta \to d\theta$ when $d\theta \to 0$,

$$|\mathbf{n}'_t| = \lim_{d\theta \to 0} \left| \frac{d\theta}{dt} \right| \leq \frac{1}{(f(x(t)) - \|x(t) - \ell(t)\|)} \leq \frac{1}{(f(x(t)) - \rho f(x))}$$

since

$$\|x(t) - \ell(t)\| \leq \|x - \ell(t)\| \leq \rho f(x).$$

Also,

$$\|x(t) - x\| \leq \|x(t) - \ell(t)\| + \|x - \ell(t)\| \leq 2\rho f(x).$$

By Lipschitz Continuity Lemma (1.3) $f(x(t)) \geq (1 - 2\rho)f(x)$. Therefore,

$$|\mathbf{n}_t'| \leq \frac{1}{(1-3\rho)f(x)} \text{ and } \angle(\mathbf{n}_x, \mathbf{n}_y) \leq \frac{\rho}{1-3\rho}.$$

□

3.1.3 Edge and triangle normals

In section 2.1, we have seen that edges joining nearby points on a curve are almost parallel to the tangents at the endpoints of the edge. Similar results also hold for surfaces. But, the measurement of size is done by the size of the smallest circumscribing balls. In fact, a triangle connecting three nearby points on a surface may lie almost perpendicular to surface. However, if its diametric ball is small compared to the local feature sizes at its vertices, it has to lie almost parallel to the surface. For an edge, its length is the same as the diameter of its diametric ball. Therefore, a small edge lies almost parallel to the surface. In essence if an edge or a triangle has a circumscribing ball that is small, it must lie flat to the surface. We quantify these claims in the next two lemmas.

Lemma 3.4 (Edge Normal.) *For an edge pq, the angle $\angle_a(\vec{pq}, \mathbf{n}_p)$ is more than* $\frac{\pi}{2} - \arcsin\frac{\|p-q\|}{2f(p)}$.

PROOF. Consider the two medial balls sandwiching the surface Σ at p. The point q cannot completely lie in any of these two balls as they are empty of points from Σ. So, the smallest angle pq makes with $\mathbf{n}_p$ cannot be smaller than the angle pq makes with $\mathbf{n}_p$ when q is on the boundary of any of these two balls. In this case let θ be the angle between pq and the tangent plane at p. It is clear that (see Figure 3.5)

$$\begin{aligned}\sin\theta &= \frac{\|p-q\|}{2\|m-p\|}\\ &\leq \frac{\|p-q\|}{2f(p)}.\end{aligned}$$

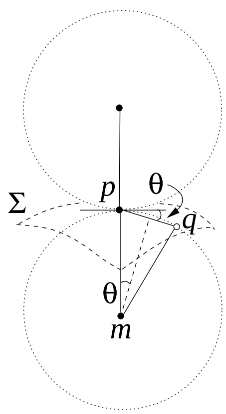

Figure 3.5: Illustration for Edge Normal Lemma.

Therefore,

$$\begin{aligned}\angle_a(\vec{pq}, \mathbf{n}_p) &= \frac{\pi}{2} - \theta \\ &> \frac{\pi}{2} - \arcsin\frac{\|p-q\|}{2f(p)}.\end{aligned}$$

□

It follows immediately from Edge Normal Lemma (3.4) that small edges make a large angle with the surface normals at the vertices. For example, if pq has a length less than $\rho f(p)$, the angle $\angle_a(\vec{pq}, \mathbf{n}_p)$ is more than $\frac{\pi}{2} - \arcsin\frac{\rho}{2}$.

Next consider a triangle $t = pqr$ where p is the vertex subtending a maximal angle in pqr. Let R_{pqr} denote the radius of the diametric ball of pqr.

Lemma 3.5 (Triangle Normal.) *If $R_{pqr} \leq \frac{f(p)}{\sqrt{2}}$, the angle between the normal $\mathbf{n}_{pqr}$ of pqr and the normal $\mathbf{n}_p$ is no more than*

$$\arcsin\frac{R_{pqr}}{f(p)} + \arcsin\left(\frac{2}{\sqrt{3}}\sin\left(2\arcsin\frac{R_{pqr}}{f(p)}\right)\right).$$

PROOF. Consider the medial balls $B = B_{m,\ell}$ and $B' = B_{m',\ell'}$ that are tangent to Σ at p. Let D be the diametric ball of t; refer to Figure 3.6. The radius of D is R_{pqr}. Let C and C' be the disks in which D intersects B and B' respectively. The line normal to Σ at p passes through m, the center of

B. This normal line makes an angle less than α with the normals to the planes of C and C', where

$$\begin{aligned}\alpha &\leq \arcsin \frac{R_{pqr}}{\|p-m\|} \text{ (since } R_{pqr} \leq radius_C) \\ &\leq \arcsin \frac{R_{pqr}}{f(p)}.\end{aligned}$$

The above angle bound also applies to the plane of C', which implies that the planes of C and C' make a wedge, say W, with an acute dihedral angle no more than 2α.

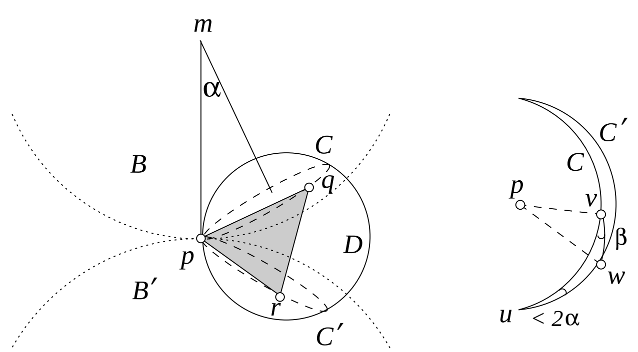

Figure 3.6: Illustration for Triangle Normal Lemma (3.5). The two great arcs on the right picture are the intersections of the unit sphere with the planes of C and C'.

The other two vertices q, r of t cannot lie inside B or B'. This implies that t lies completely in the wedge W. Consider a polyhedral cone at p inside the wedge W formed by the three planes; π_t, the plane of t, π, the plane of C and π', the plane of C'. A unit sphere centered around p intersects the cone in a spherical triangle uvw, where u, v and w are the points of intersections of the lines $\pi \cap \pi'$, $\pi_t \cap \pi$ and $\pi_t \cap \pi'$ respectively with the unit sphere. See the picture on right in Figure 3.6. Without the loss of generality, assume that the angle $\angle uvw \leq \angle uwv$. We have the following facts. The arc length of wv, denoted $|wv|$, is at least $\pi/3$ since p subtends the largest angle in t and t lies completely in the wedge W. The spherical angle $\angle vuw$ is less than or equal to 2α. We are interested in the spherical angle $\beta = \angle uvw$ which is also the acute dihedral angle between the planes of t and C. By standard sine laws in spherical geometry, we have $\sin\beta = \sin|uw| \frac{\sin \angle vuw}{\sin|wv|} \leq \sin|uw| \frac{\sin 2\alpha}{\sin|wv|}$. If $\pi/3 \leq |wv| \leq 2\pi/3$, we have $\sin \pi/3 \geq \sqrt{3}/2$ and hence

$\beta \leq \arcsin(\frac{2}{\sqrt{3}} \sin 2\alpha)$. For the range $2\pi/3 < |wv| < \pi$, we use the fact that $|uw| + |wv| \leq \pi$ since $\angle vuw \leq 2\alpha < \pi/2$ for $R_{pqr} \leq \frac{f(p)}{\sqrt{2}}$. So, in this case $\frac{\sin |uw|}{\sin |wv|} < 1$. Thus, $\beta \leq \arcsin \frac{2}{\sqrt{3}} \sin 2\alpha$.

The normals to t and Σ at p make an acute angle at most $\alpha + \beta$ proving the lemma. □