1.3 Voronoi and Delaunay geometry

Voronoi diagrams and Delaunay triangulations are important geometric data structures that are built on the notion of 'nearness'. Many differential properties of curves and surfaces are defined on local neighborhood. Voronoi diagram provides a tool to approximate these neighborhoods in the discrete domain.

Voronoi diagrams. Let P be a set of points in the plane $\mathbb{R}^2$. The Voronoi diagram of P denoted V_P is a collection of Voronoi cells V_p for each point $p \in P$, where

$$V_p = \{x \in \mathbb{R}^2 \mid \|x - p\| \leq \|x - q\| \text{ for any } q \in P.\}$$

In words, V_p is the set of all points in the plane that are closer or at least equidistant to p than any other point in P. Figure 1.8 (a) shows a Voronoi diagram of a point set in the plane.

For any two points p, q the set of points closer to p than q are demarked by the perpendicular bisector of the segment pq. This means the Voronoi cell $V_p \in V_P$ is the intersection of halfplanes determined by the perpendicular bisectors between p and each other point $q \in P$. Implication of this observation is that each Voronoi cell is a convex polygon since the intersection of convex sets remains convex.

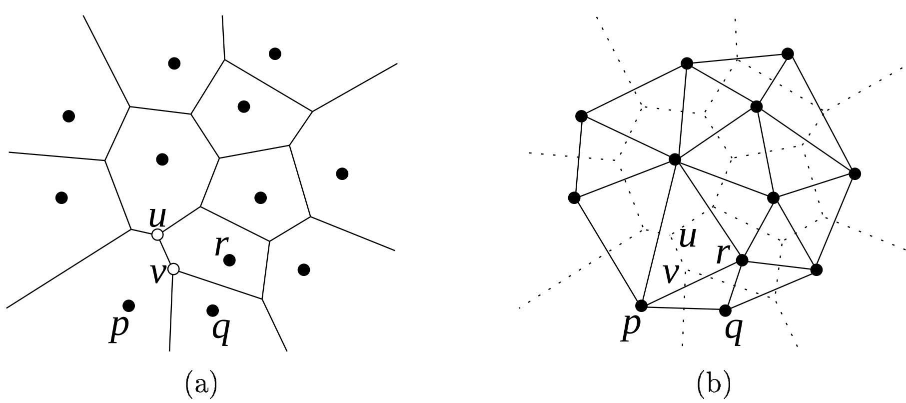

Figure 1.8: Voronoi diagram and Delaunay triangulation of a point set in the plane.

Voronoi cells have *Voronoi faces* of different dimensions. A Voronoi face F of dimension k is the intersection of $3 - k$ Voronoi cells. This means a k-dimensional Voronoi face for $k < 2$ is the set of all points that are equidistant from $3-k$ points in P. A zero dimensional Voronoi face, called *Voronoi vertex* is equidistant from three points in P, whereas an one dimensional Voronoi face, called *Voronoi edge* contains points that are equidistant from two points in P. A Voronoi cell is a two dimensional Voronoi face. In Figure 1.8 (a) u and v are Voronoi vertices and uv is a Voronoi edge.

Some of the Voronoi cells may be unbounded with unbounded edges. It is a straightforward consequence of the definition that a Voronoi cell V_p is unbounded if and only if p is on the boundary of the convex hull of P. In Figure 1.8 (a) V_p and V_q are unbounded and p and q are on the convex hull boundary.

Delaunay triangulation. There is an associated *dual* structure to Voronoi diagram V_P, called the *Delaunay triangulation* denoted D_P. Formally, we define D_P as a simplicial complex where

$$D_P = \{\sigma \mid \bigcap V_p \neq \emptyset \text{ where } p \text{ is any vertex of } \sigma.\}$$

In words, $k+1$ points in P form a Delaunay k-simplex in D_P if their Voronoi cells have nonempty intersection. We know that $k+1$ Voronoi cells meet in a $(2-k)$-dimensional Voronoi face. So, each k-simplex in D_P is dual to a $2-k$ dimensional Voronoi face. Thus, each Delaunay triangle pqr in D_P is dual to a Voronoi vertex where V_p, V_q and V_r meet; each Delaunay edge pq is dual to a Voronoi edge shared by Voronoi cells V_p and V_q, and each vertex p is dual to its corresponding Voronoi cell V_p. In Figure 1.8 (b), the Delaunay triangle pqr is dual to the Voronoi vertex v and the Delaunay edge pr is dual to the Voronoi edge uv.

Dual Voronoi vertices of Delaunay triangles are equidistant from its three vertices. This means that the circumcenter of a Delaunay triangle coincides with the dual Voronoi vertex. It implies that no point from P can lie in the interior of the circumscribing ball of a Delaunay triangle. These balls are called *empty*. The converse also holds.

Property 1 (Triangle emptiness.) *A triangle is in the Delaunay triangulation if and only if its circumscribing ball is empty.*

The Triangle Emptiness Property of Delaunay triangles also implies a similar emptiness for Delaunay edges. Clearly, each Delaunay edge has an empty circumscribing ball passing through its endpoints. It turns out that the converse is also true, that is, any edge pq with an empty cisrcumscribing ball must also be in the Delaunay triangulation. To see this, grow the empty ball of pq always keeping p, q on its boundary. If it never meets any other point from P, the edge pq is on the convex hull boundary of P and is in the Delaunay triangulation since V_p and V_q has to share an edge extending to infinity. Otherwise, when it meets a third point, say r from P, we have an empty circumscribing ball passing through p, q, r. By the Triangle Emptiness Property pqr must be in the Delaunay triangulation and hence the edge pq.

Property 2 (Edge emptiness.) *An edge pq is in the Delaunay triangulation D_P if and only if pq has an empty circumscribing ball.*

Restricted Voronoi diagrams. When the input point set P is a sample of a curve or a surface Σ, we have structures imposed by Voronoi diagram V_P on Σ. It turns out that this diagram plays an important role in reconstructing Σ from P. Formally, a cell in the restricted Voronoi diagram $V_{P,\Sigma}$ is defined as the intersection of a Voronoi cell in V_P with Σ, i.e.,

$$V_{P,\Sigma} = \{V_{p,\Sigma} = V_p \cap \Sigma \mid p \in P\}.$$

Similar to the Voronoi diagram, we can define *restricted Voronoi faces* as the intersection of restricted Voronoi cells. They can also be viewed as the intersection of Voronoi

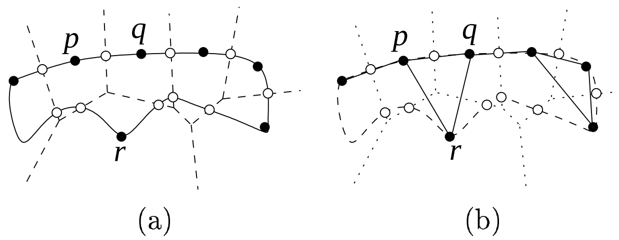

Figure 1.9: (a) Restricted Voronoi diagram, (b) restricted Delaunay triangulation for a point set on a curve.

faces with Σ. In Figure 1.9 (a) the white circles represent restricted Voronoi faces of dimension zero. The curve segments between them are restricted Voronoi faces of dimension one which are restricted Voronoi cells in this case. Notice that the restricted Voronoi cell $V_{p,\Sigma}$ consists of two curve segments whereas $V_{q,\Sigma}$ consists of a single curve segment.

Restricted Delaunay triangulation. As with Voronoi diagrams we can define a simplicial complex dual to the restricted Voronoi diagrams. A k-simplex in this dual complex, called *restricted Delaunay triangulation*, is defined with $k+1$ vertices, R, where

$$\bigcap V_{p,\Sigma} \neq \emptyset, \text{ for } p \in R.$$

In Figure 1.9 the picture in (b) shows the restricted Delaunay triangulation for the restricted Voronoi diagram in (a). The vertex p is connected to q and r in the restricted Delaunay triangulation since $V_{p,\Sigma}$ meets both $V_{q,\Sigma}$ and $V_{r,\Sigma}$. However the triangle pqr is not in the triangulation since $V_{p,\Sigma}$, $V_{q,\Sigma}$ and $V_{r,\Sigma}$ do not meet at a point.

Three dimensions. We chose the plane to explain the concepts of Voronoi diagram and Delaunay triangulation in the previous discussions. However, these concepts extend to arbitrary dimensions. We will mention these extensions for three dimensions which will be important for later expositions.

Vornoi cells of a point set P in $\mathbb{R}^3$ are three dimensional convex polytopes some of which are unbounded. There are four types of Voronoi faces; Voronoi vertex, Voronoi edge, Voronoi facet and Voronoi cell in increasing order of dimenison sarting with zero and ending with three. Four Voronoi cells meet at a Voronoi vertex which is equidistant from four points in P. Three Voronoi cells meet at a Voronoi edge and two Voronoi cells meet at a Voronoi facet.

The Delaunay triangulation of P contains four types of simplices dual to each of the four types of Voronoi faces. The vertices are dual to Voronoi cells, the Delaunay edges are dual to Voronoi facets, the Delaunay triangles are dual to Voronoi edges, and Delaunay tetrahedra are dual to Voronoi vertices. The circumscribing ball of each tetrahedron is empty. Conversely, any tetrahedron with empty circumscribing ball is in the Delaunay triangulation. Further, each Delaunay triangle and edge has an empty ball that passes through its vertices. Conversely, an edge or a triangle belongs to the Delaunay triangulation if there exists an empty ball passing through their vertices.

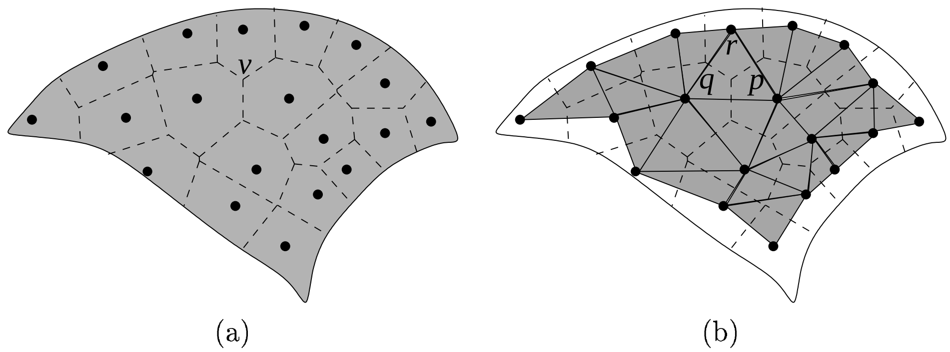

Figure 1.10: (a) Restricted Voronoi diagram, and (b) restricted Delaunay triangulation for a sample on a surface.

We can define the restricted Voronoi diagram and its dual restricted Delaunay triangulation for a point sample on a curve or surface in $\mathbb{R}^3$ in the same way as we did in $\mathbb{R}^2$. Figure 1.10 shows restricted Voronoi diagram and its dual restricted Delaunay triangulation for a set of points on a surface. The triangle pqr is in the restricted Delaunay triangulation since $V_{p,\Sigma}$, $V_{q,\Sigma}$ and $V_{r,\Sigma}$ meet at a common point v.