Lecture 23: Voronoi diagram and Delaunay triangulation [1]

We have seen that Bézier and B-spline curves and surfaces need an initial control net. Even the subdivision surfaces refines only an initial net. The question is how to obtain this initial net. This brings up the *reconstruction* problem. Given a set of samples from a curve or a surface we want to compute a piecewise linear curve or surface that interpolates through the sample points and approximate the sampled curve or surface.

We will need Voronoi diagram and Delaunay triangulation for reconstructing curves and surfaces. We will concentrate on curve reconstruction in two dimensions. So, we review Voronoi diagram and Delaunay triangulation in two dimensions.

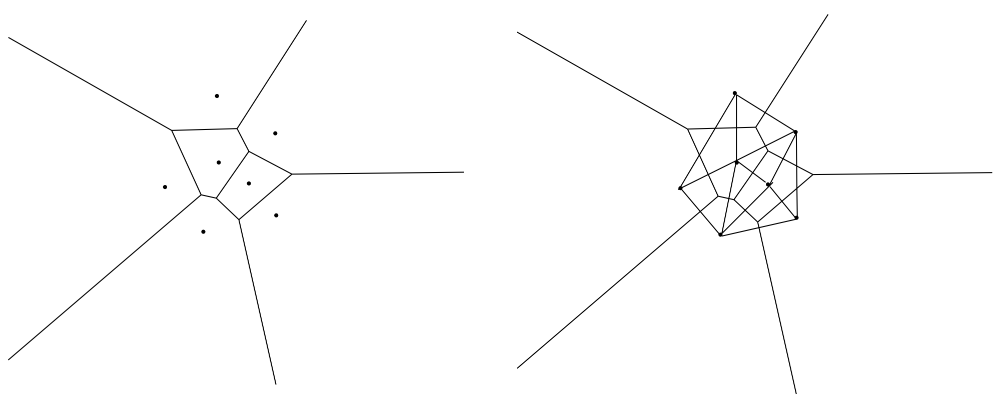
Voronoi Diagram

Given a set of points in the plane, Voronoi diagram determines the region of influence of each point in such a way that the regions decompose the plane.

Let P be a set of n points in the plane. For each point $p \in P$, the Voronoi region V_p for p is defined as the set of all points that are at least as close to p as any other point in P.

$$V_p = \{x \in R^2 || xp| \leq |xq|, \forall q \in P\}$$

See Figure for an illustration.



Consider the half-plane of points at least as close to p as to q: $H_{pq} = \{x \in R^2 || xp| \leq |xq|\}$. The Voronoi region of p is the intersection of half-planes H_{pq}, for all $q \in P - \{p\}$. It follows that V_p is a convex polygonal region. We list some important properties of the Voronoi diagram under the general position assumption.

1. Each Voronoi cell is convex.
2. Unbounded Voronoi cells correspond to vertices on the convex hull.
3. Voronoi cells are interior-wise disjoint.
4. Two Voronoi cells either do not intersect, or intersect in an edge called Voronoi edge.
5. Three Voronoi cells that meet pairwise along Voronoi edges also meet in a vertex called Voronoi vertex.

[1]Note by Tamal K. Dey, Ohio State U.

6. A Voronoi edge lies on the perpendicular bisector of the two generating points.

7. A Voronoi vertex is equidistant from the three generating points and thus is the circumcenter of the triangle spanning these three points.

8. There are n Voronoi cells and at most $3n - 6$ Voronoi edges.

Delaunay triangulation

There is a dual diagram of Voronoi diagram which produces a triangulation of the point set P. This is called the Delaunay triangulation of P which we denote as D_p. See Figure for an illustration.

Two points p, q are connected with an edge in D_p if and only if V_p and V_q share an edge. Three points p, q, r are connected with a triangle if and only if V_p, V_q and V_r share a Voronoi vertex. We list some of the important properties of the Delaunay triangulation.

1. If pqr is a triangle in D_p, then the circumcenter of pqr is a Voronoi vertex. It follows that the circumcircle of pqr is empty of any other point from P.

2. Conversely, if there is a triangle pqr whose circumcircle is empty of any other point from P, then pqr must be in D_p.

3. Each edge in D_p must have a circumcircle that is empty. Conversely, any edge that has one empty circumcircle must be in D_p.

4. D_p forms a planar graph with n points. Thus, it can have at most $3n - 6$ edges and $2n - 4$ triangles.

There are some nice properties of Delaunay triangulations. Let us call an edge pq is *locally Delaunay* if the two triangles pqr and pqs incident on pq satisfy the property that the circumcircle of pqr does not have s inside. It can be shown that the circumcircle of pqs also does not contain r in that case.

CLAIM. A triangulation T of a point set P is Delaunay if and only if each edge is locally Delaunay.

In general, Delaunay triangulations produce well shaped triangles. Indeed, among all possible triangulations of P, D_p maximizes the minimum angle in a triangle.