Lecture 22: CSG Models and Euler Operators [1]

CSG models

In CSG models, a solid is built up from simpler solids with boolean operations and simple transformations. But, before we explain the model we define the boolean operations taking degeneracies into account.

Regularized boolean operations

Consider the Figure 11.33 in the book. The intersection of A and B produces a non-manifold solid with a hanging edge. The problem is caused by degeneracies in the intersection. We would like to define the boolean operations that are closed with respect to manifold-property, i.e., if we take two manifold solids, the resulting solid should also be a manifold.

For this we use regularized boolean operations. Let $C = A \cap B$. The interior iS of a solid S is defined as the set of points that have an open neighborhood homeomorphic to a open ball. We first consider iC and then take its closure to define the resulting solid C^*. The closure will add the points that have neighborhood in iC. These points will constitute the boundary bC^* of C^*. If we do this, the resulting solid in Figure **??** will be a manifold. The regularized intersection, union, and difference are denoted as $\cap^*, \cup^*, -^*$ respectively. Now let us see how we can implement a regularized intersection for two polygons A and B.

We can write

$$\begin{aligned} C &= A \cap B \\ &= (bA \cup iA) \cap (bB \cup iB) \\ &= (bA \cap bB) \cup (iA \cap bB) \cup (bA \cap iB) \cup (iA \cap iB) \end{aligned}$$

It can be observed that $iC^* = iA \cap iB$. Next we must determine bC^*, where $bC^* = valid(bA \cap bB)$. It can be shown that

$$iA \cap bB \subset bC^*$$
$$bA \cap iB \subset bC^*$$

We must now analyze $(bA \cap bB)$ to determine which of its subsets are valid subsets of bC^*. Observe that these sets are neither interior to A, nor to B. We make a distinction among these points depending on their neighborhoods. A point on the vertical segment (Figure 11.35) has no point in the neighborhood that lies inside both of A and B. But, a point on the horizontal segment has this property. This distinguishes segment 1 not to be in the $valid_i(bA \cap bB)$. So, we define $valid_i(bA \cap bB)$ as the set of boundary points that have a neighborhood point lying in the interior of both of A and B. With these definitions we have

$$C^* = valid_i(bA \cap bB) \cup (iA \cap bB) \cup (bA \cap iB) \cup (iA \cap iB)$$

Think about how to implement the regularized union and difference operations.

[1]Note by Tamal K. Dey, Ohio State U.

Constructive Solid Geometry

Complex solids are made out of simpler ones using boolean operations and simple transformations. We assume that the boolean operations are regularized. The model is represented with a binary tree, where the leaf nodes are simple primitive shapes, and the branch nodes are the set operations. See Figure 11.56 in the book. The boundary representation is computed from the CSG representation by boundary evaluators which compute the boolean operations.

Euler operators

In solid modeling system often objects are modified to create new objects by adding, deleting faces, edges and vertices. To check if the resulting solid is a valid object, we need some rules. Euler's equation comes handy in this case. To apply the formula, certain conditions must be satisfied.

1. All faces are simply connected, i.e., they are topological disks.
2. Each edge must join exactly two faces and terminate at a vertex.
3. At least three edges must meet at each vertex.

We can apply $V - E + F = 2 - 2G$ to verify if the modifications to an object is valid. Here we assume that the topology of the solid should not change by the modifications. If we allow the faces to be non-simple, i.e., have holes then the Euler formula is modified as

$$V - E + F - H + 2G = 2C$$

where H is the number of holes in the faces and G denotes the number of "through holes", C denotes the number of separate, disjoint solids. One can always impose the condition that each face is simple since any polygon can always be decomposed into simple faces, for example triangles (try to prove this statement).