Lecture 2: Bézier Curves I [1]

Named after P. Bézier these parametric curves *approximates* a set of points called *control* points. Unlike Hermite interpolants these do not interpolate the points. The shape of the curve can be controled by moving the control points. But, this does not give the predictable control as the change in shape is global rather than local. We will see later that B-splines achieve this.

Parametric form:

Let $\mathbf{p}_0, ..., \mathbf{p}_n$ be a set of n control points forming the vertices of a control polygon. The general equation is:

$$\mathbf{p}(u) = \Sigma_{i=0}^{n} \mathbf{p}_i f_i(u), u \in [0,1]$$

The basis functions f_i's are chosen so that the following holds.

- The curve goes through $\mathbf{p}_0$ and $\mathbf{p}_n$.
- The tangent at $\mathbf{p}_0$ and $\mathbf{p}_n$ are given by $\mathbf{p}_1 - \mathbf{p}_0$ and $\mathbf{p}_n - \mathbf{p}_{n-1}$.
- We can require higher order derivatives at the endpoints be controled by appropriate number of points.
- $f_i(u)$ is symmetric with respect to u and $1-u$.

Basis functions:

Bernstein polynomials $B_{i,n}$ are chosen for basis functions $f_i(u) = B_{i,n}$ where

$$B_{i,n}(u) = \binom{n}{i} u^i (1-u)^{n-i}$$

These are nth degree polynomials.

$$B_{0,2} = (1-u)^2$$
$$B_{1,2} = 2u(1-u)$$
$$B_{2,2} = u^2$$

So, with three control points we have

$$\mathbf{p}(u) = (1-u)^2 \mathbf{p}_0 + 2u(1-u)\mathbf{p}_1 + u^2 \mathbf{p}_2$$

Similarly, one can get cubic Bézier curve using four control points.

$$\mathbf{p}(u) = (1-u)^3 \mathbf{p}_0 + 3u(1-u)^2 \mathbf{p}_1 + 3u^2(1-u)\mathbf{p}_2 + u^3 \mathbf{p}_3$$

[1] Note by Tamal K. Dey

Matrix forms:

The cubic curve can be written as:

$$\mathbf{p}(u) = [(1 - 3u + 3u^2 - u^3)\ (3u - 6u^2 + 3u^3)\ (3u^2 - 3u^3)\ u^3][\mathbf{p}_0\ \mathbf{p}_1\ \mathbf{p}_2\ \mathbf{p}_3]^T$$

or

$$\mathbf{p}(u) \quad = \quad \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{p}_0 & \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix}^T$$

Compactly, we can write $\mathbf{p}(u) = \mathbf{U}\mathbf{M_b}\mathbf{P}$ where $\mathbf{U}$ is the u-vector, $\mathbf{M_b}$ is the co-efficient matrix and $\mathbf{P}$ is the point vector above. We will use this matrix form later in subdivisions.

Affine invariance:

Bézier curves are invariant under affine transformation, i.e., translation, rotation, scaling, or shear.
This means that the following should be true. Let

$$\mathbf{p}(u_i) = \Sigma_{i=0}^{n}\mathbf{p}_i B_{i,n}(u_i)$$

Apply the affine transformation $\mathbf{A}$ so that $\mathbf{p}'(u_i) = \mathbf{A}\mathbf{p}(u_i)$. This point is same as the point obtained after transforming the control points and then using the approximation, i.e., $\mathbf{p}'(u_i) = \Sigma_{i=0}^{n}\mathbf{p}'_i B_{i,n}(u_i)$ where $\mathbf{p}'_i = \mathbf{A}\mathbf{p}_i$.

A Bézier curve is also invariant under affine reparameterization. So, if $u \in [0, 1]$ and $v \in [a, b]$ then using $u = \frac{(v-a)}{(b-a)}$ we have

$$\Sigma_{i=0}^{n}\mathbf{p}_i B_{i,n}(u) = \Sigma_{i=0}^{n}\mathbf{p}_i B_{i,n}\left(\frac{v-a}{b-a}\right).$$

Convex hull property

Observe that $\Sigma_{i=0}^{n} B_{i,n}(u) = (u + (1-u))^n = 1$ and also $B_{i,n}(u) \geq 0$ for $u \in [0, 1]$. Thus the points on $\mathbf{p}(u)$ is a convex combination of the control points, and thus reside inside their convex hull.