

Lecture 12: Bézier Surface ¹

Equation

Just as the curve Bézier surface has a control polyhedron. Points on a Bézier surface is given by a simple extension of the Bézier curves

$$\mathbf{p}(u, w) = \sum_{i=0}^m \sum_{j=0}^n \mathbf{p}_{i,j} B_{i,m}(u) B_{j,n}(w), \quad u, w \in [0, 1]$$

where $\mathbf{p}_{i,j}$ are the vertices of the control polyhedron.

We again observe that

$$\sum_{i=0}^m \sum_{j=0}^n B_{i,m}(u) B_{j,n}(w) = \sum_{i=0}^m B_{i,m}(u) = 1$$

So, the Bézier surface patch resides inside the convex hull of the vertices of its control polyhedron.

Matrix form

We can express a Bézier equation for a surface in a matrix form $\mathbf{U} \mathbf{M} \mathbf{P} \mathbf{M}' \mathbf{W}^T$. The dimension of the matrices \mathbf{M} and \mathbf{M}' depends on the dimension of the control point matrix. If \mathbf{P} is a $(m+1) \times (n+1)$ matrix then the matrix form with proper dimensions reads

$$\mathbf{U}_{1 \times m} \mathbf{M}_{m \times m} \mathbf{P}_{m \times n} \mathbf{M}'_{n \times n} \mathbf{W}_{n \times 1}^T$$

Let us consider a bicubic patch. It requires a 4×4 array of control points:

\mathbf{p}_{00}	\mathbf{p}_{01}	\mathbf{p}_{02}	\mathbf{p}_{03}
\mathbf{p}_{10}	\mathbf{p}_{11}	\mathbf{p}_{12}	\mathbf{p}_{13}
\mathbf{p}_{20}	\mathbf{p}_{21}	\mathbf{p}_{22}	\mathbf{p}_{23}
\mathbf{p}_{30}	\mathbf{p}_{31}	\mathbf{p}_{32}	\mathbf{p}_{33}

The equation is given by

$$\mathbf{p}(u, w) = \begin{bmatrix} (1-u)^3 & 3u(1-u)^2 & 3u^2(1-u) & u^3 \end{bmatrix} \mathbf{P} \begin{bmatrix} (1-w)^3 \\ 3w(1-w)^2 \\ 3w^2(1-w) \\ w^3 \end{bmatrix}$$

We can deduce that only four corners $\mathbf{p}_{00}, \mathbf{p}_{03}, \mathbf{p}_{30}, \mathbf{p}_{33}$ lie on the surface. There are four boundary curves, $\mathbf{p}_{0w}, \mathbf{p}_{1w}, \mathbf{p}_{u0}, \mathbf{p}_{u1}$ given by $u = 0, u = 1, w = 0$ and $w = 1$ respectively. For example, the boundary curve \mathbf{p}_{0w} is generated with the control polygon given by $\mathbf{p}_{00}, \mathbf{p}_{01}, \mathbf{p}_{02}, \mathbf{p}_{03}$:

$$\mathbf{p}(0, w) = [\mathbf{p}_{00} \ \mathbf{p}_{01} \ \mathbf{p}_{02} \ \mathbf{p}_{03}] \begin{bmatrix} (1-w)^3 \\ 3w(1-w)^2 \\ 3w^2(1-w) \\ w^3 \end{bmatrix}$$

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We can generate any curve for a constant value of one parameter on the patch by generating the control points as follows:

Let $w = w_i$ be the parameter value for which we want to generate the curve on the patch. The control points for it is given by:

$$\begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \mathbf{P} \begin{bmatrix} (1 - w_i)^3 \\ 3w_i(1 - w_i)^2 \\ 3w_i^2(1 - w_i) \\ w_i^3 \end{bmatrix}$$

Occasionally we would like to create a patch that is defined by a \mathbf{P} matrix that is not square. For example, one can have a \mathbf{P} matrix of dimension 5×3 .

$$\mathbf{p}(u, w) = \begin{bmatrix} (1 - u)^4 & 4u(1 - u)^3 & 6u^2(1 - u)^2 & 4u^3(1 - u) & u^4 \end{bmatrix} \\ \times \begin{bmatrix} \mathbf{p}_{00} & \mathbf{p}_{01} & \mathbf{p}_{02} \\ \mathbf{p}_{10} & \mathbf{p}_{11} & \mathbf{p}_{12} \\ \mathbf{p}_{20} & \mathbf{p}_{21} & \mathbf{p}_{22} \\ \mathbf{p}_{30} & \mathbf{p}_{31} & \mathbf{p}_{32} \\ \mathbf{p}_{40} & \mathbf{p}_{41} & \mathbf{p}_{42} \end{bmatrix} \begin{bmatrix} (1 - w)^2 \\ 2w(1 - w) \\ w^2 \end{bmatrix}$$

The advantage of a five point boundary curve over a four point boundary curve is that a change in the third and middle point of the curve does not affect the slope at either end. Thus, patch shape can be changed without changing the end slopes, hence maintaining continuity with adjacent patches.