Lecture 11: Introduction to Surface Equations [1]

Implicit surfaces

An equation of the form $f(x,y,z)=0$ is an implicit equation of a surface. In addition if $f(x,y,z)$ is a polynomial in x,y,z we have a polynomial surface with implicit equation

$$\Sigma_{i,j,k} a_{i,j,k} x^i y^j z^k = 0$$

The degree of the surface is the maximum degree of a term, $i+j+k$. If we solve one variable in terms of the other, say, z, we get explicit equation $z=f(x,y)$ of the same surface. A rational parametric form of a surface is expressed as $z=f(u,v), x=g(u,v), y=h(u,v)$ where f,g,h are polynomials. Converting a parametric form to an implicit form is called *implicitization* and it is possible to implicitize every parametric surface.

Quadric surfaces

A quadric polynomial surface is expressed with an implicit equation of degree 2. A general form of a quadric surface is:

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2Fzx + 2Gx + 2Hy + 2Jz + K = 0$$

In matrix form one can write: $\mathbf{PQP}^T$ where $\mathbf{P} = [x\ y\ z\ 1]$ and

$$\mathbf{Q} = \begin{bmatrix} \mathrm{A} & \mathrm{D} & \mathrm{F} & \mathrm{G} \\ \mathrm{D} & \mathrm{B} & \mathrm{E} & \mathrm{H} \\ \mathrm{F} & \mathrm{E} & \mathrm{C} & \mathrm{J} \\ \mathrm{G} & \mathrm{H} & \mathrm{J} & \mathrm{K} \end{bmatrix}$$

Except for some degenerate cases, all quadric surfaces that has explicit eqaution $z=f(x,y)$ of degree 2 must be one of the following three types:

$$\begin{aligned} z &= x^2/a^2 + y^2/b^2 \ \textit{elliptic paraboloid} \\ z &= x^2/a^2 - y^2/b^2 \ \textit{hyperbolic paraboloid} \\ y^2 &= 4ax \ \textit{parabolic cylinder} \end{aligned}$$

It should be noted that a parametric surface of degree 2 may correspond to an explicit surface of degree more than 2. For example,

$$\begin{aligned} x &= u \\ y &= u^2 + v \\ z &= v^2 \end{aligned}$$

An explicit equation of this surface is $z=(y-x^2)^2$ which is of degree 4 and its degree cannot be lower.

[1] Note by Tamal K. Dey, Ohio State U.

Curvatures

For each point on a surface we can define its curvatures that tells us how fast the surface is bending locally. Consider the tangent plane at a point p on the surface. The surface normal at that point is defined as the normal to the tangent plane. Consider the planes passing through the line supporting the normal. These planes intersect the surface in open curves in an open neighborhood of p. Among these curves, the two curves with maximum and minimum curvatures are of most importance to us. These two curvatures with signs are called *principal curvatures* at p. The larger (absolute value) of the two, say κ_1 is called the maximum curvature and the smaller of the two, say κ_2 is called the minimum curvature at p. The *Gaussian curvature* at p is defined as $\kappa = \kappa_1\kappa_2$. The arithmetic mean of the two curvatures is called the *mean curvature* at p.

Now one can observe that hyperbolic paraboloid has a Gaussian curvature negative at the origin. Elliptic paraboloid has Gaussian curvature positive at the origin. The cylinderical paraboloid has Gaussian curvature 0 at the origin.

Any surface locally can be approximated with a degree 2 explicit surface and thus each point is either elliptical, or hyperbolic or parabolic.

Parametric surfaces

We will talk about it when we cover Bézier and B-spline surfaces.