Lecture 10: Basic Surface Topology [1]

Manifolds

In three dimensions the topology of surfaces becomes an important factor in modeling. We will mean a manifold when we talk about surfaces in 3D.

A 2-manifold $S \in R^k$ is a topological space such that each point in S is *homeomorphic* to a 2-disk. Homeomorphism is a continuous function defined between two spaces which is bijective and also has a continuous inverse. For example, a square in plane is homeomorphic to a disk, a "surface patch" in 3D as we will call it is also homeomorphic to a 2-disk.

A 2-manifold may be embedded in R^3 meaning that it has no self-intersection. Or, it might be *immersed* in R^3 in which case there is no self-intersection.

Some examples of 2-manifolds are spheres, torus, double torus. We know that some of the surfaces as we know might have boundaries. For example, a "surface patch" has a boundary. Our definition of 2-manifolds do not allow such "surface patch". So, we need another definition. We define 2-manifold *with boundary* as a topological space such that each point has a neighborhood homeomorphic to a half-disk. A sphere with a hole cut out is a 2-manifold with boundary. The points which has only half-disk neighborhood constitute the boundary. In general, the boundary of a 2-manifold is a 1-manifold that is a closed curve.

Classification of surfaces

We will calll 2-manifolds as *surfaces* and 2-manifolds with boundary as *surfaces with boundary.*

Surfaces in 3D have a nice charaterization upto topology. A 2-manifold is either a sphere or a join of one or more torus. A join of two tori form the double torus, and in general join k tori form a surface called k-tori.

The *genus* of a surface is the minimum number of cut required to make it flat or a disk. For example, a torus needs two cuts one along the equator, and one along meridian to make it flat. Only one cut makes it a cylinder which again can be cut to make the rectangle. Conversely, a rectangle can be made into a torus by identifying the opposite edges of a rectangle. Actually, this process can be generalized for arbitrary 2-manifold surfaces. A genus-g surface can be obtained by identifying appropriate edges of a $4g$-gon. The sphere is a special case whose genus is 0.

A similar charaterization of surfaces with boundary is also possible. The characterization takes into account the genus of the surface and the number of boundaries.

Euler Characteristic

There is a combinatorial characterization of surfaces that is also sometimes useful in modeling. We assume only two types of surface patches that a surface in decomposed into. They are either triangular or rectangular. In each case we assume the surface patches join nicely to form a *complex.*, i.e., any two of the surface patches either do not meet, or meet in an edge or a vertex of both.

Let v, e and f denote the number of vertices, edges and faces (patches) of a surface complex. The quantity $v-e+f$ is called the Euler characteristic of the surface which is a topological invariant. It means that any two surfaces that are homeomorphic must have the same Euler characteristic. For

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[1] Note by Tamal K. Dey

example, the boundary of a tetrahedron is homeomorphic to a sphere, and its Euler characteristic is $4 - 6 + 4 = 2$ which is same as the Euler characteristic of a cube boundary $8 - 12 + 6 = 2$. In general, the genus-g surface has an Euler characteristic of $2 - 2g$. Thus, the torus has Euler characteristic 0.

We will often want a rectangular net on a surface with each vertex having degree four. Such a net is essential for generating Bézier or B-spline surfaces. Using Euler characteristic we can show that there is no rectangular net that can span a sphere and has degree four at each vertex. If this were possible we would have:

$$\begin{aligned} 2e &= 4v \; from\ degree\ consideration \\ 2e &= 4f \; each\ face\ rectangular \\ v - e + f &= 2 \; Eulercharacteristic \\ or,\ \frac{1}{2}e - e + \frac{1}{2}e &= 2,\ an\ impossibility \end{aligned}$$

Thus, it is always difficult to fit a Bézier or B-spline surfaces on sphere.

However, a torus will admit such a net because its Euler characteristic is indeed 0 which is required by the first two equations above. In fact, it is only the torus among all surfaces without boundary that admits a rectangular net. There are many surfaces with boundary that admits a rectangular net where the degree of a vertex on the boundary is 3. For eaxmple, a disk admits such a rectangular net. All surfaces admit triangular net with no restriction on degree.