2.2 Correctness

First, we show that edges joining a sample point p and its nearest neighbor is correct.

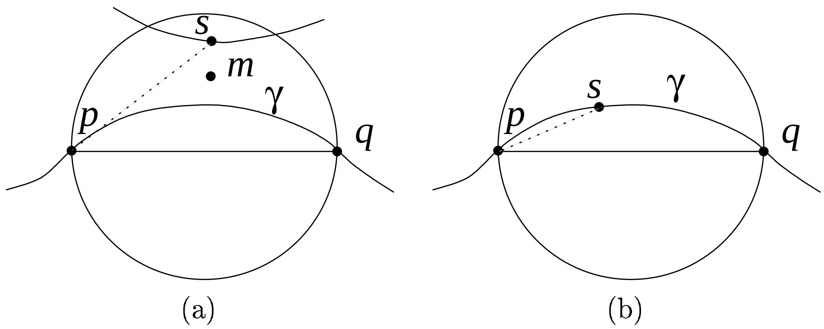

Figure 2.4: Diametric ball of pq intersects Σ in (a) two components, (b) single component.

Lemma 6 (Neighbor.) *Let $p \in P$ be any sample point and q be its nearest neighbor. The edge pq must be correct for $\varepsilon \leq \frac{1}{3}$.*

PROOF. Consider the ball B with pq as diameter. If B intersects Σ in more than one component, it must contain a medial axis point by Feature Ball Lemma. See Figure 2.4(a). This means $\|p-q\| > f(p)$. A correct edge ps satisfies $\|p-s\| \leq \frac{2\varepsilon}{1-\varepsilon} f(p)$ by Small Edge Lemma. Thus, for $\varepsilon \leq \frac{1}{3}$ we have $\|p-s\| < \|p-q\|$, a contradiction to the fact that q is the nearest neighbor to p.

Consider the other case as in Figure 2.4(b) when B intersects Σ in one curve segment, say γ. If pq is not correct, γ must contain a sample point, say s, between p and q inside B. Again, we reach a contradiction as $\|p-s\| < \|p-q\|$. □

To show that edges connecting a sample point to its half neighbors are also correct, we need the following result which says that correct edges incident to a sample point cannot subtend an angle less than $\frac{\pi}{2}$ at that point.

Lemma 7 (Angle.) *Let pq and pr be two correct edges incident to p. We must have $\angle qpr \geq \pi - 2\arcsin\frac{\varepsilon}{1-\varepsilon}$.*

PROOF. Consider the two medial balls sandwiching Σ at p as in Figure 2.5. Since q and r cannot lie inside the two balls, the angle $\angle qpr$ reaches its minimum when q and r lie on the boundary of a medial ball.

In that case,

$$\angle qpr \geq \pi - 2\alpha$$

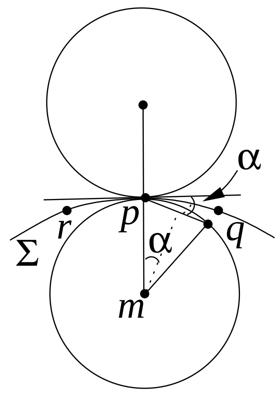

Figure 2.5: Illustration for Angle Lemma.

and

$$\begin{aligned}\alpha &\leq \arcsin\left(\frac{\|p-q\|}{2}/\|m-p\|\right)\\ &\leq \arcsin\left(\frac{\frac{2\varepsilon}{1-\varepsilon}f(p)}{2}/f(p)\right)\\ &\leq \arcsin\frac{\varepsilon}{1-\varepsilon}.\end{aligned}$$

The Lemma follows immediately. □

Lemma 8 (Half Neighbor.) *An edge pq where q is a half neighbor of p is correct when $\varepsilon \leq \frac{1}{3}$.*

PROOF. Let r be the nearest neighbor of p. According to the definition $\mathbf{t}_{pq}$ makes an angle more than $\frac{\pi}{2}$ with $\mathbf{t}_{pr}$.

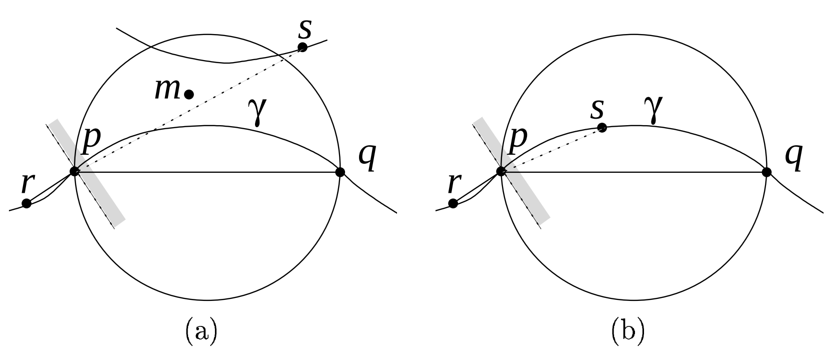

Figure 2.6: (a) Diametric ball of pq intersects Σ in (a) more than one component, (b) a single component.

If pq is not correct, consider the correct edge ps incident to p other than pr. By Angle Lemma $\mathbf{t}_{ps}$ also makes an angle more than $\frac{\pi}{2}$ with $\mathbf{t}_{pr}$ for $\varepsilon \leq 1/3$. We show that s is closer to p than q contradicting q being the half neighbor of p.

Consider the ball B with pq as a diameter. If this ball intersects Σ in more than one component as in Figure 2.6(a), the ball B would contain a medial axis point, and thus $\|p-q\| > f(p)$. On the other hand $\|p-s\| \leq \frac{2\varepsilon}{1-\varepsilon}$ by Small Edge Lemma. We reach a contradiction as $\|p-s\| < \|p-q\|$ for $\varepsilon \leq \frac{1}{3}$. So, assume that B intersects Σ in one component as in Figure 2.6(b). This means one of the curve segments between p and q is completely contained in B. Since pq is not a correct edge, s must be on this curve segment. It implies $\|p-s\| < \|p-q\|$ reaching a contradiction again. □

Theorem 1 CURVERECON *computes all and only correct edges.*

PROOF. Step 4 and 5 assure that all edges joining samples points to their nearest and half neighbors are computed as output. These edges are correct by Neighbor Lemma and Half Neighbor Lemma. Also, there is no other correct edges since each sample point can only be incident to exactly two correct edges.