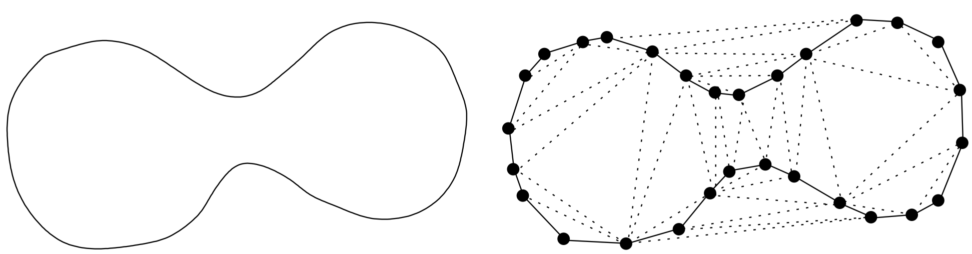
Chapter 2

Curve Reconstruction

2.1 Reconstruction

A single smooth curve that we will deal with is defined by a function $\sigma : [0,1] \to \mathbb{R}^2$ where σ is one-to-one except at the endpoints where $\sigma(0) = \sigma(1)$, and has a continuous non-zero first derivative. When we refer to a curve Σ in the plane, we actually mean the graph of one or more such functions. By definition Σ does not self-intersect though can have multiple components each of which is closed, i.e., without any end point.

Let P be a set of sample points on Σ. An edge connecting two sample points, say p and q, is called *correct* if p and q are two consecutive points on Σ. Otherwise, it is *incorrect*. The goal of *curve reconstruction* is to compute a piecewise linear curve consisting of all and only correct edges. In Figure 2.1(c) all solid edges are correct and dotted edges are incorrect.



(a) (b)

Figure 2.1: (a) A smooth curve, (b) its reconstruction from a sample shown with solid edges.

Neighbors. We present an algorithm to reconstruct Σ which assumes that the input P is an ε-sample of Σ. This algorithm is based on the concept of nearest neighbors. A point $p \in P$ is called a nearest neighbor of $q \in P$ if there is no other point $r \in P \setminus p, q$ with $\|q-r\| < \|q-p\|$. Observe that p being a nearest neighbor of q does not necessarily mean that q is a nearest neighbor of p.

We first observe that edges that connect nearest neighbors in P must be correct edges if P is an ε-sample. But, all correct edges do not connect nearest neighbors. Figure 2.2 shows all edges that connect nearest neighbors. The missing correct edges in this example connect points that are not nearest neighbors. However, these correct edges connect points that are *half neighbors* as we define below.

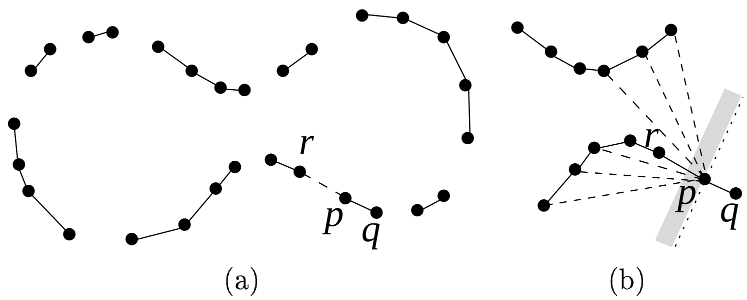


Figure 2.2: (a) Only nearest neighbor edges do not reconstruct a curve, (b) half neighbor edges fill up the gaps such as pr.

Let pq be an edge connecting p to its nearest neighbor q and $\mathbf{t}_{pq}$ be an unit vector from p to q. Consider the halfplane H with $q \notin H$ and $\mathbf{t}_{pq}$ being normal to its boundary. The nearest neighbor to p in the set $H \cap P$ is called its *half neighbor*. In Figure 2.2(b), r is the half neighbor of p. We will show later that two correct edges incident to a sample point must connect it to its nearest and half neighbors.

Algorithm. The above discussion immediately suggest an algorithm for curve reconstruction. But, we need efficient algorithms to compute nearest neighbor and half neighbor for each sample point. The Delaunay triangulation D_P turns out to be useful for this computation as all correct edges are Delaunay if P is an ε-sample. First, we need to show that correct edges are small in length.

Lemma 4 (Small Edge.) *For any correct edge pq we must have $\|p-q\| < \frac{2\varepsilon}{1-\varepsilon} f(p)$ when $\varepsilon < 1$.*

PROOF. Consider the curve segment γ over which p and q are adjacent on Σ. Let the perpendicular bisector of pq intersect this curve segment at x. Let r be the nearest sample point to x. We claim that r is p or q. Suppose it is not. Then, the ball B with center x and with radius $\|r-x\|$ must intersect Γ in two components since there is no sample point between p and q on the curve segment γ; see Figure 2.3. By Feature Ball Lemma B must have a medial axis point which implies that $\|r-x\| > f(x)$. But, $\|r-x\| \leq \varepsilon f(x)$ by ε-sampling condition. We reach a contradiction for $\epsilon < 1$.

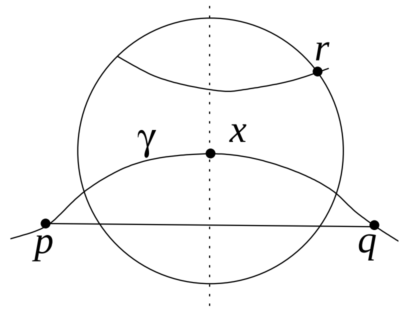


Figure 2.3: Illustration for Small Edge Lemma.

It follows that p and q are the nearest sample points to x. Applying the ε-sampling condition and Feature Translation Lemma we get

$$\begin{aligned} \|p-q\| &\leq 2\|p-x\| \\ &\leq 2\varepsilon f(x) \\ &\leq \frac{2\varepsilon}{1-\varepsilon} f(p). \end{aligned}$$

□

Lemma 5 (Delaunay Subset.) *A correct edge pq must be Delaunay for $\varepsilon \leq \frac{1}{3}$.*

PROOF. Consider the diametric ball of the edge pq. If this ball is empty, pq is Delaunay by Edge Emptiness Lemma. Otherwise, let r be another sample point contained inside B. Since the curve segment γ over which p and q are adjacent cannot contain any other sample point, r must be on the complement curve segment $\Sigma \setminus \gamma$. This means *Int* B intersects Σ in at least two components. Therefore, B contains a medial axis point, say m, by Feature Ball Lemma. We have $\|p-q\| > \|p-m\| \geq f(p)$. But, $\|p-q\| \leq \frac{2\varepsilon}{1-\varepsilon} f(p)$ by Small Edge Lemma. We reach a contradiction when $\frac{2\varepsilon}{1-\varepsilon} \leq 1$ or $\varepsilon \leq \frac{1}{3}$. □

Delaunay Subset Lemma implies that, for each sample point p, it is sufficient to check only the Delaunay edges to determine correct edges. In the example of Figure 2.1 all Delaunay edges are shown out of which the solid ones are correct. We check all edges incident to p in D_P and determine the shortest edge connecting it to its nearest neighbor, say q. Next, we check all other edges incident to p which make an angle more than $\frac{\pi}{2}$ with pq at p and choose the shortest among them. This second edge connects p to its half neighbor. This entire computation can be done in time proportional to the number of edges incident to p. Since the sum of the number of incident edges over all vertices in the Delaunay triangulation is $O(n)$ where $|P| = n$, correct edge computation takes only $O(n)$ time once D_P is computed. Delaunay triangulation of a set of n points in the plane can be computed in time $\Theta(n \log n)$ which implies that the following curve recosntruction algorithm takes $O(n \log n)$ time.

CurveRecon (P)
compute D_P;
$E = \emptyset$;
for each $p \in P$
    compute the shortest edge pq in D_P;
    compute the shortest edge pr so that $\angle pqr > \frac{\pi}{2}$;
    $E = E \cup \{pq, pr\}$;
endfor
output E.