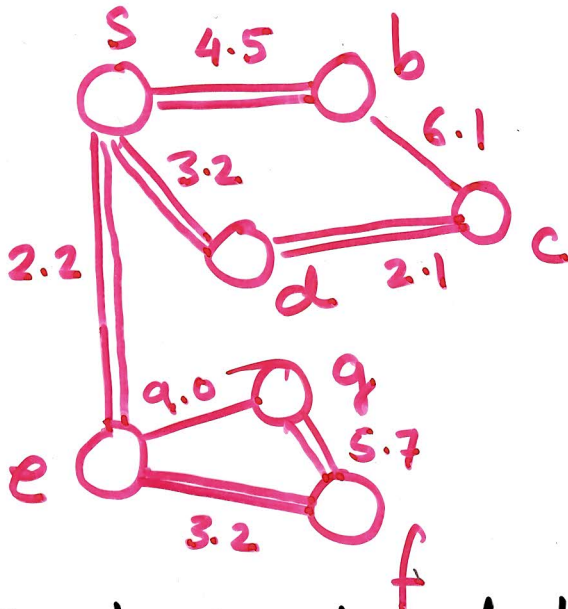


Single Source Shortest Paths ①

Given a graph (connected) $G=(V,E)$ with weights on edges and source vertex $s \in V$, find the shortest path from s to any other vertex $v \in V$.

Ex.



$$\text{length}(s,b,c) = 10.6$$

$$\text{length}(s,d,c) = 5.3$$

$$\delta(s,c) = 5.3$$

The graph produced by the shortest paths form a spanning tree.

Why no cycle?



There are two paths from s to v . Assuming only one path to v is created if there are two or more paths of equal lengths, c cannot be present.

Dijkstra's Algorithm.

Starting from the source s , we grow a shortest path tree and gradually expand it. For all vertices v that are not yet

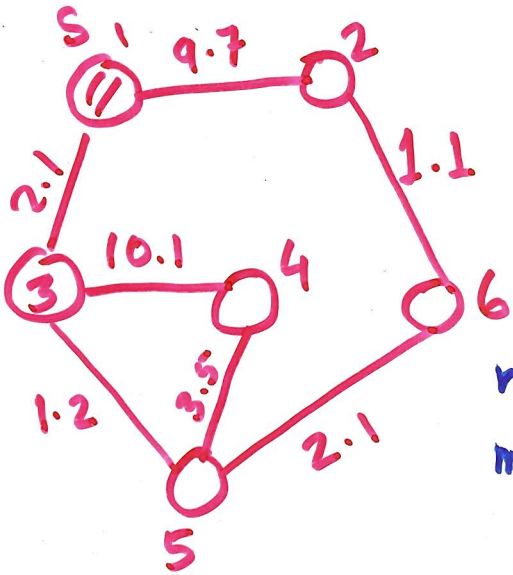
in the tree. We maintain a p -value that reflects the length of the path from s to v using only vertices in the tree except v . (2)

The Algorithm.

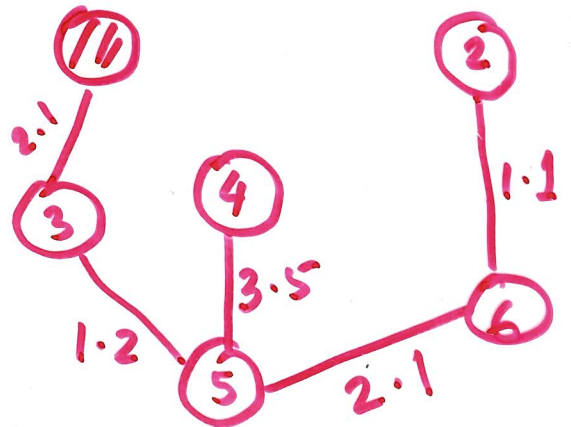
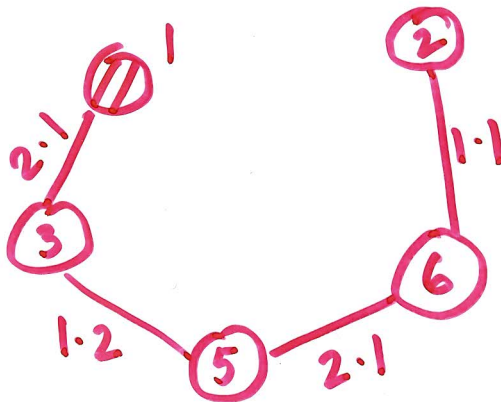
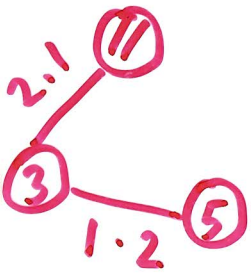
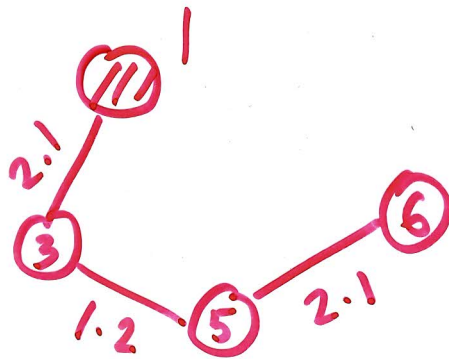
```
PQ :=  $\emptyset$ ; for  $i := 1$  to  $n$  do
  if  $i \neq k$  then  $V[i].p := \infty$ 
  else  $V[i].p := 0$ ;  $V[i].\pi := \text{nil}$ ;
endif
add  $i$  with priority  $V[i].p$  to PQ
endfor
```

```
while PQ  $\neq \emptyset$  do
   $i := \text{Extractmin}(PQ)$ ;  $t := V[i].\text{adj}$ ;
  while  $t \neq \text{nil}$  do
     $j := t.v$ ;
    if  $j \in PQ$  and  $V[i].p + w(i, j) \leq V[j].p$  then
       $\Delta := V[i].p - (V[i].p + w(i, j))$ ;
      Decrease-key(PQ,  $j, \Delta$ );
       $V[j].\pi := i$ ;
    endif
  endwhile
endwhile
```

Example.



	1	2	3	4	5	6
	0	∞	∞	∞	∞	∞
min 9.7		9.7	2.1	∞	∞	∞
min 2.1		9.7	2.1	12.2	3.3	∞
min 3.3		9.7		6.8	3.3	5.4
min 5.4		6.5		6.8		5.4
min 6.5		6.5		6.8		
min 6.8				6.8		



Time Complexity:

(4)

If PQ is an array : $O(n^2)$

If PQ is an ordinary heap: $O((n+m)\log n)$

If PQ is a Fibonacci heap: $O(m + n\log n)$

n Extract-min
 m Decrease-key

Correctness: Let S be the set of vertices marked so far and r be the vertex being included into S .

Inductive Hypothesis:

(i) For any vertex $i \in S$, $V[i].p = \delta(s, i)$

(ii) For any vertex $i \notin S$, $V[i].p$ is the length of the shortest path to i that goes through only the vertices in S except i itself.

Base Case $S = \{s\}$, $V[s].p = 0$ and $V[i].p = w(s, i)$
When i is adjacent to s and
 $V[i].p = \infty$ for all other i .

(i) & (ii) are satisfied trivially.

