Flow Networks

A <u>flow network</u> is a directed graph $G=(V,E)$ in which each edge $(u,v)\in E$ has a nonnegative <u>capacity</u> $c(u,v)\geq 0$. If $(u,v)\notin E$, we assume $c(u,v)=0$. There are two distinguished vertices <u>source</u> s, <u>sink</u> t.

We also assume G is connected. Therefore $|E|\geq|V|-1$.

<u>Flow</u>: $f: V\times V \to \mathbb{R}$ is a real-valued function.

1. for $\forall\ u,v\in V$, $f(u,v)\leq c(u,v)$.
2. " " " , $f(u,v)=-f(v,u)$
3. " " $u\in V-\{s,t\}$, $\sum_{v\in V} f(u,v)=0$

The value of a flow is $|f|=\sum_{v\in V} f(s,v)$.

Maxflow problem is to find $\max |f|$.

Some properties of flow

$$f(X,Y) = \sum_{x\in X}\sum_{y\in Y} f(x,y) \text{ for two sets}$$

$X, Y \subseteq V$.

1. $f(X,X) = 0$
2. $f(X,Y) = -f(Y,X)$
3. $f(X\cup Y, Z) = f(X,Z) + f(Y,Z)$
4. $f(Z, X\cup Y) = f(Z,X) + f(Z,Y)$.

Ford-Fulkerson Algorithm

Uses the concept of Residual capacity:

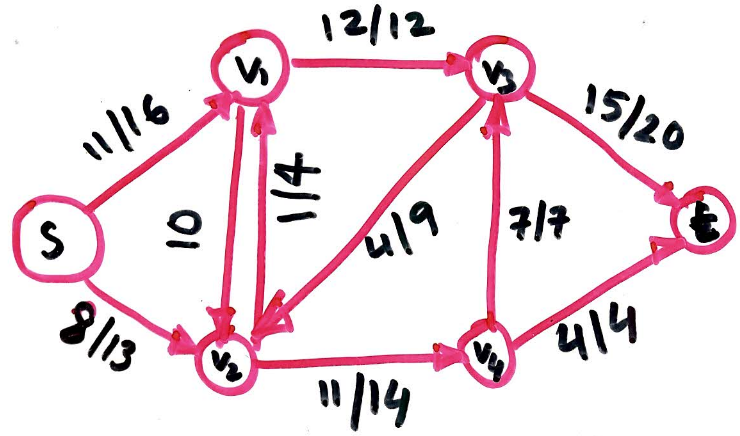
$$c_f(u,v) = c(u,v) - f(u,v).$$

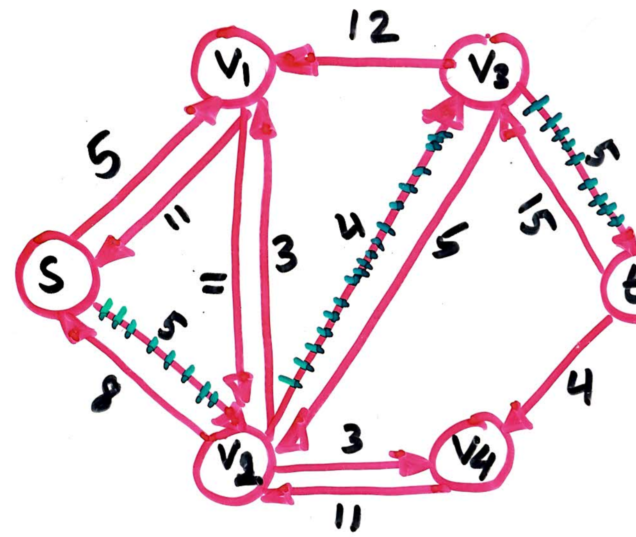
Given a flow network $G=(V,E)$ and a flow f, the residual network is

$G_f = (V, E_f)$ where

$E_f = \{(u,v)\in V\times V \mid c_f(u,v) > 0\}$.

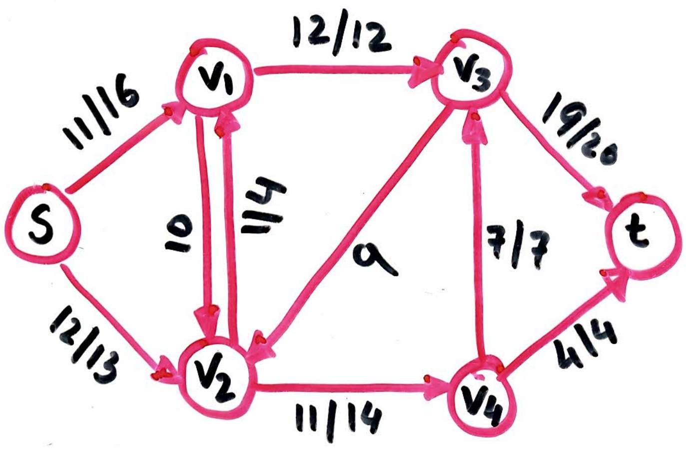
Example.

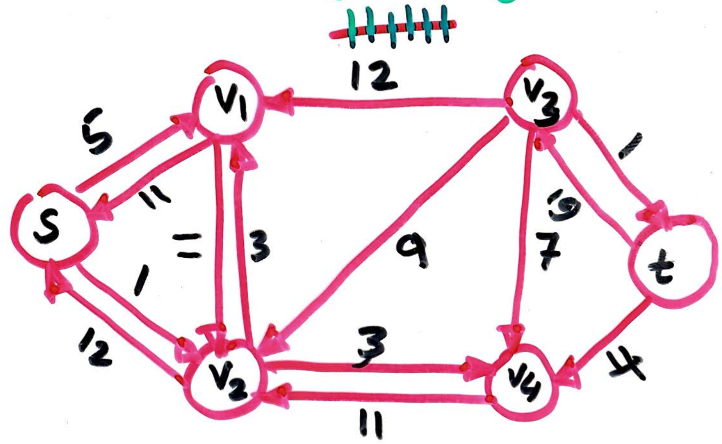




Residual Network

Augmenting path





No augmenting path

Augmenting path. A path p from s to t in the residual network G_f. The residual capacity of p is given by

$$c_f(p) = \min \{ c_f(u,v) \mid (u,v) \in p \}$$

Algorithm

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Ford-Fulkerson(G, s, t)
   for each edge (u,v) ∈ E
         do f[u,v] := 0
            f[v,u] := 0
   endfor
   while there exists a path p from s to t
         in G_f do
         c_f(p) := min{c_f(u,v) | (u,v) ∈ p}
         for each edge (u,v) ∈ p
             f[u,v] := f[u,v] + c_f(p)
             f[v,u] := -f[u,v]
         endfor
   endwhile
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If capacities are integers, the above algorithm runs in $O(|E||f^*|)$ time where $|f^*|$ is the max flow. Each iteration in the while loop can be implemented in $O(|E|)$ time increasing the flow at least by one unit.

Max flow - Min Cut

Lemma 1

Let G_f be the residual network induced by flow f. Let f' be a flow in G_f. Then $f+f'$ is a flow in G where

$$|f+f'| = |f| + |f'|.$$

Proof.

Skew symmetry:

$$\begin{aligned}(f+f')(u,v) &= f(u,v) + f'(u,v)\\ &= -(f(v,u) + f'(v,u))\\ &= -(f+f')(v,u)\end{aligned}$$

Capacity constraint:

$$\begin{aligned}(f+f')(u,v) &= f(u,v) + f'(u,v)\\ &\le f(u,v) + c_f(u,v)\\ &= f(u,v) + c(u,v) - f(u,v)\\ &= c(u,v)\end{aligned}$$

Flow conservation: for $\forall u \in V - \{s,t\}$

$$\begin{aligned}\sum_{v\in V} (f+f')(u,v) &= \sum_{v\in V} (f(u,v) + f'(u,v))\\ &= \sum_{v\in V} f(u,v) + \sum_{v\in V} f'(u,v)\\ &= 0 + 0 = 0\end{aligned}$$