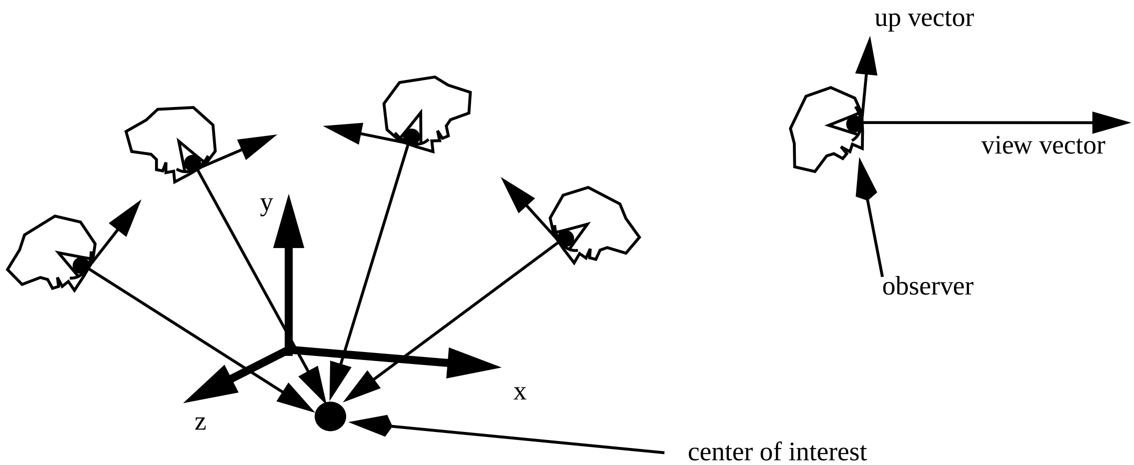
$$
\begin{aligned}
w &= COI - EYE && \text{view direction vector} \\
u &= w \times (0, 1, 0) && \text{cross product with y-axis} \\
v &= w \times u && \text{up vector}
\end{aligned}
\qquad \text{(EQ 1)}
$$


up vector
view vector
observer
y
x
z
center of interest

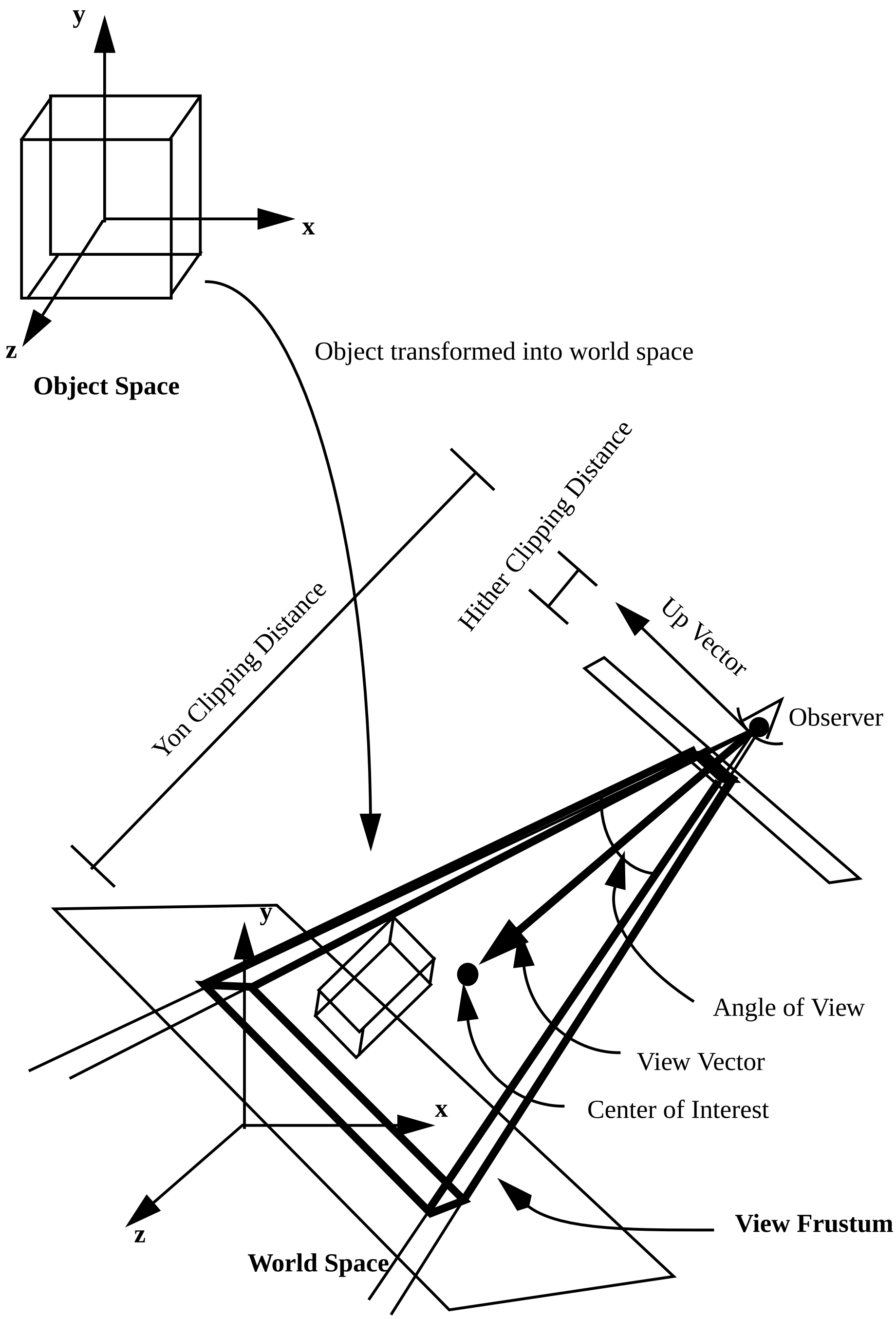


FIGURE 7. Object to world space transformation and the view frustum in world space.

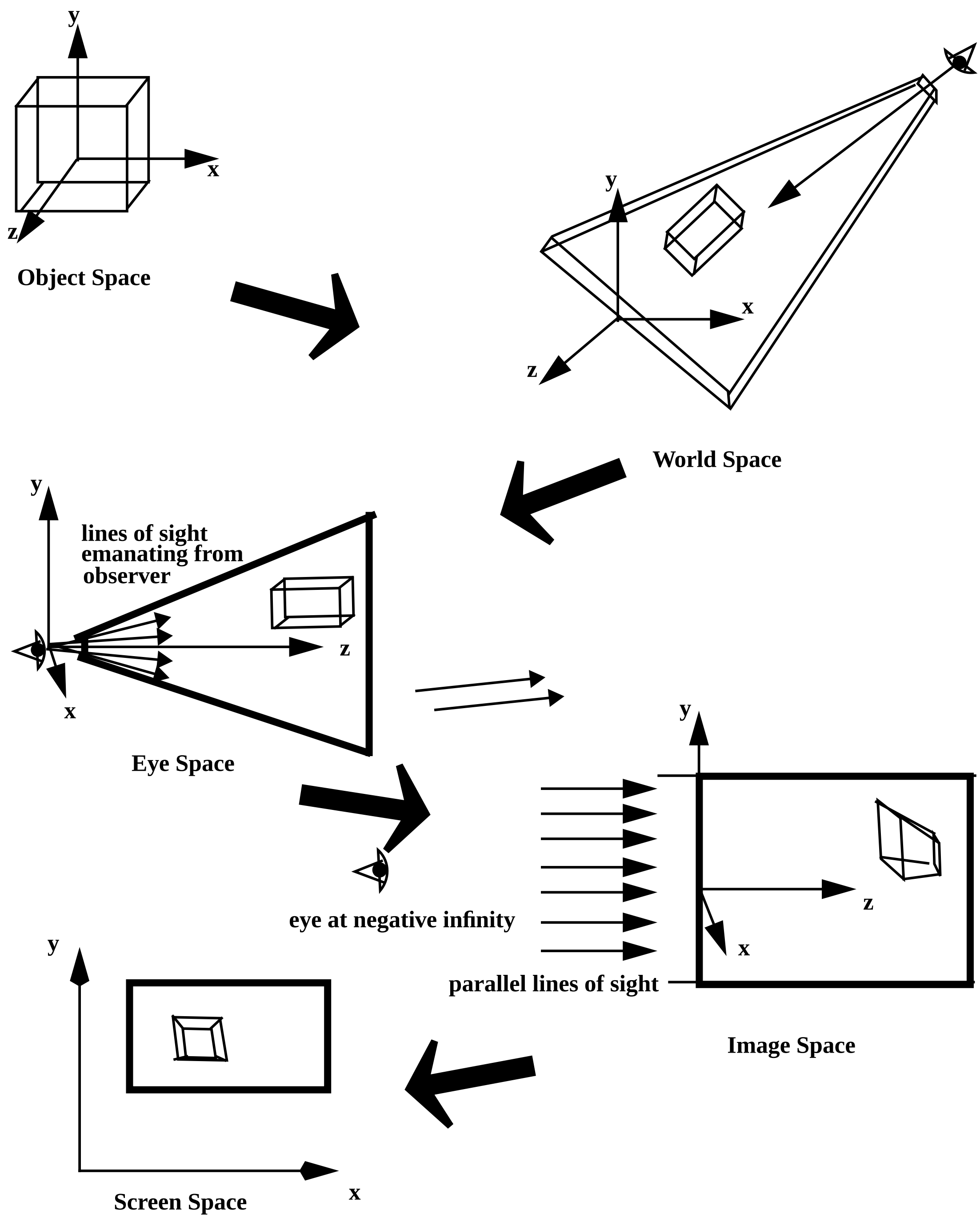


FIGURE 8. **Display pipeline showing transformation between spaces.**

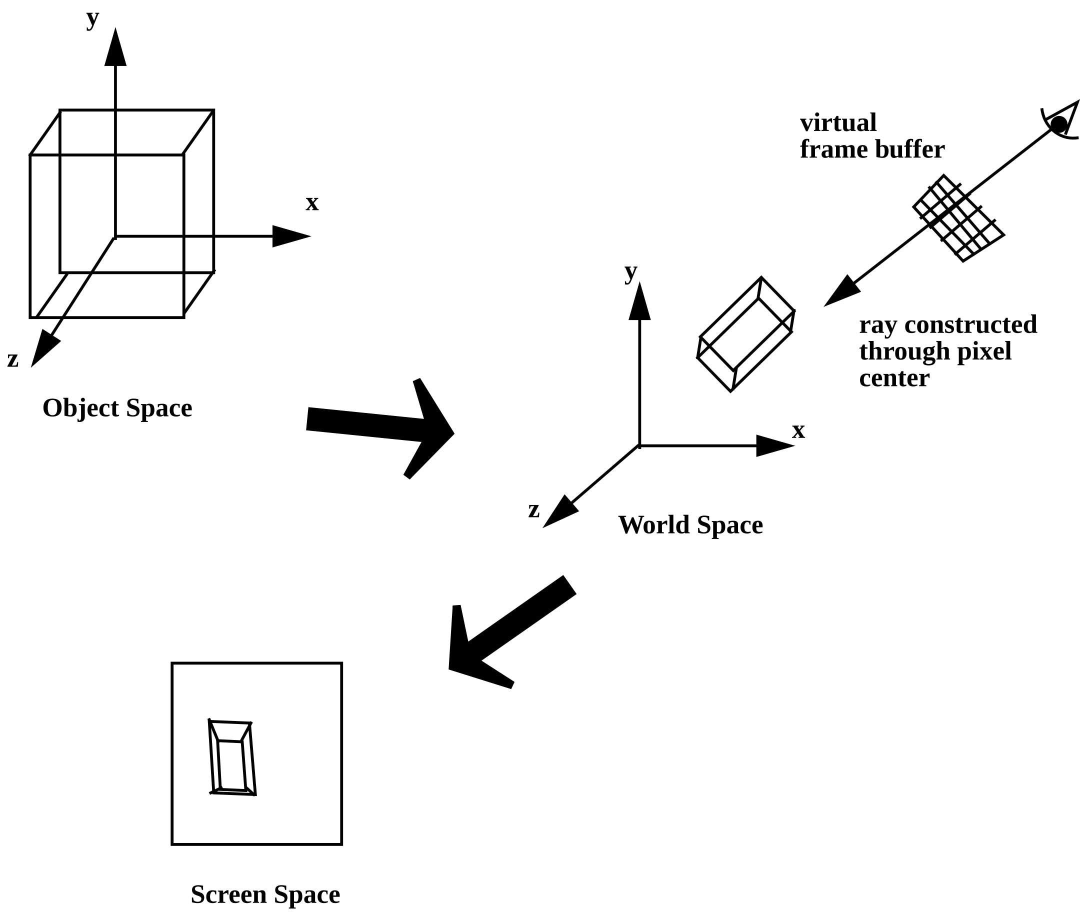


FIGURE 9. **Transformation through spaces using ray casting.**

$$\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right) = [x, y, z, w] \quad \text{(EQ 2)}$$

$$(x, y, z) = [x, y, z, 1] \quad \text{(EQ 3)}$$

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad \text{(EQ 4)}$$

$$\begin{bmatrix} x & y & z & 1 \end{bmatrix} = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(EQ 5)}$$

$$\begin{aligned} P' &= M_1 M_2 M_3 M_4 M_5 M_6 P \\ M &= M_1 M_2 M_3 M_4 M_5 M_6 \\ P' &= MP \end{aligned} \quad \text{(EQ 6)}$$

$$\begin{aligned} P' &= P M_6^T M_5^T M_4^T M_3^T M_2^T M_1^T \\ M^T &= M_6^T M_5^T M_4^T M_3^T M_2^T M_1^T \\ P' &= P M^T \end{aligned} \quad \text{(EQ 7)}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & m \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad \text{(EQ 8)}$$

$$\begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad \text{(EQ 9)}$$

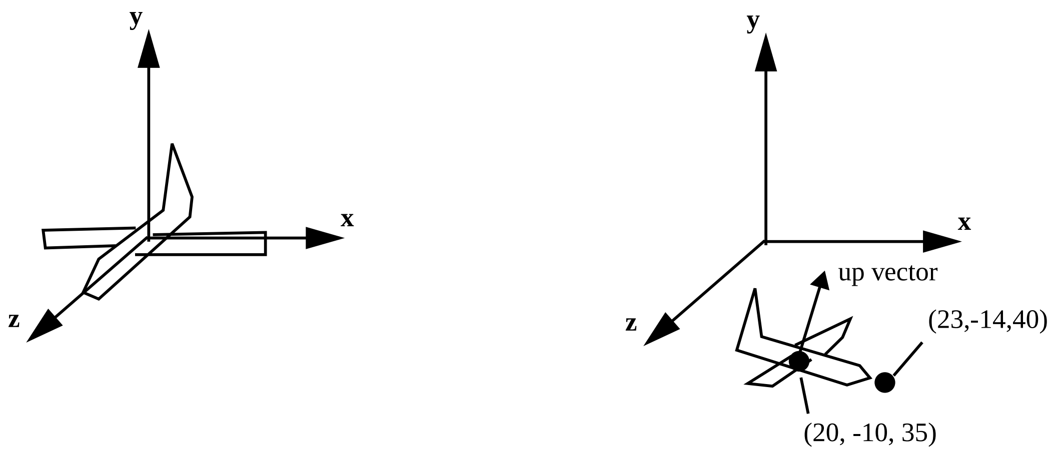
$$\begin{bmatrix} S_x \cdot x \\ S_y \cdot y \\ S_z \cdot z \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{EQ 10}$$

$$\begin{bmatrix} S \cdot x \\ S \cdot y \\ S \cdot z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ \frac{1}{S} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{S} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{EQ 11}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{EQ 12}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{EQ 13}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{EQ 14}$$



a) Object space definition

b) World space position and orientation of aircraft

FIGURE 10. Desired Position and Orientation.

.

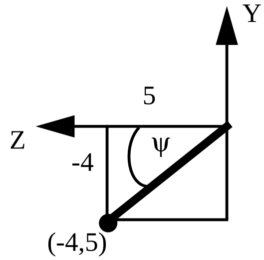


FIGURE 11. **Projection of desired orientation vector onto y-z plane.**

.

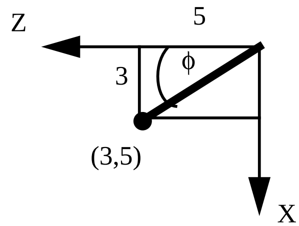
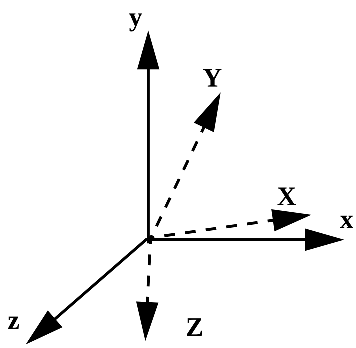


FIGURE 12. **Projection of desired orientation vector onto x-z plane.**



x,y,z - global coordinate system

X,Y,Z - deisred orientation defined by unit coordinate system

FIGURE 13. Global coordinate system and unit coordinate system to be transformed.

$$X = M \cdot x \qquad Y = M \cdot y \qquad Z = M \cdot z$$

$$\begin{bmatrix} X_x \\ X_y \\ X_z \end{bmatrix} = M \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} Y_x \\ Y_y \\ Y_z \end{bmatrix} = M \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} Z_x \\ Z_y \\ Z_z \end{bmatrix} = M \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{EQ 15}$$

$$\begin{bmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{bmatrix} = M \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{bmatrix} = M \tag{EQ 16}$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{EQ 17}$$

.

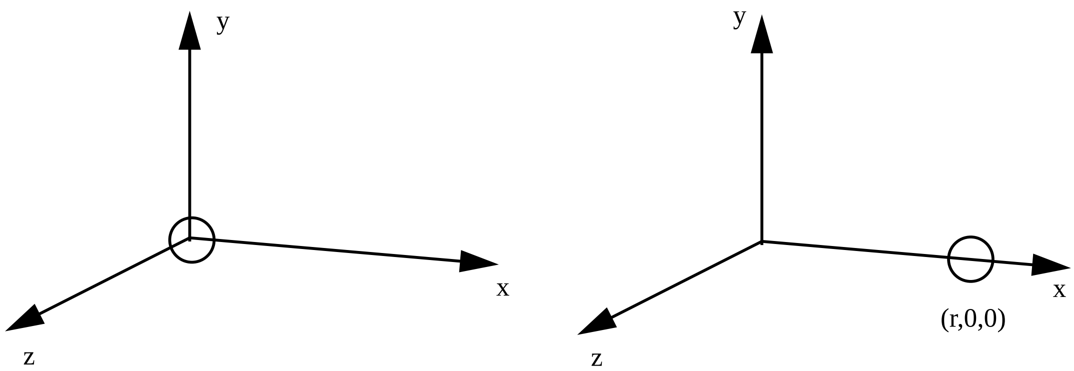
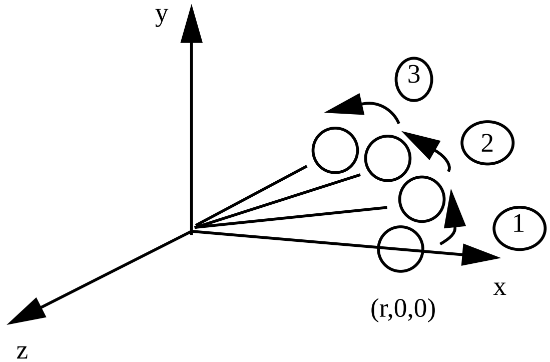
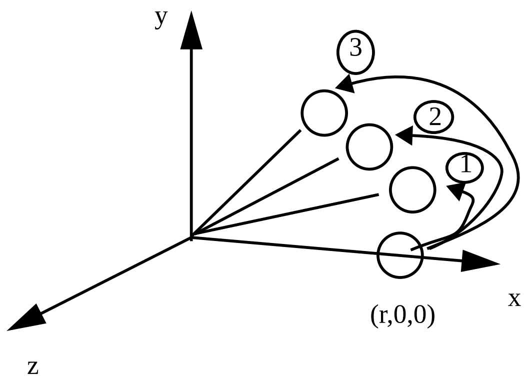


FIGURE 14. Translation of moon out to its initial position on the x-axis.



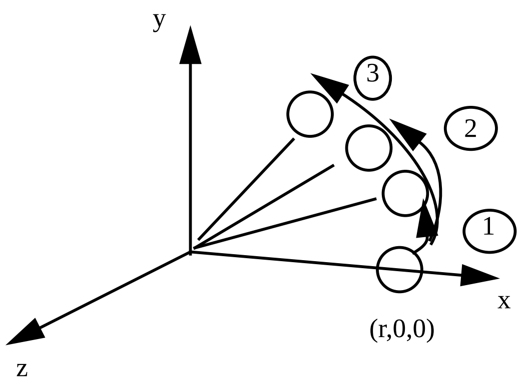

```
for each point P of the moon {
     P’ = P
}
Rdy= y-axis rotation of 5 degrees
repeat until (done) {
   for each point P’ of the moon {
        P’ = Rdy*P’
   }
   record a frame of the animation
}
```

FIGURE 15. Rotation by applying incremental rotation matrices to points.



R = identity matrix
R_{dy}= y-axis rotation of 5 degrees
repeat until (done) {
  for each point P of the moon {
    P’ = R*P
  }
  record a frame of the animation
  R = R*R_{dy}
}

FIGURE 16. **Rotation by incrementally updating the rotation matrix.**




```
y = 0
repeat until (done) {
   R = y-axis rotation matrix of ‘y’ degrees
   for each point P of the moon {
         P’ = R*P
   }
   record a frame of the animation
   y = y+5
}
```

FIGURE 17. Rotation by forming the rotation matrix new for each frame.

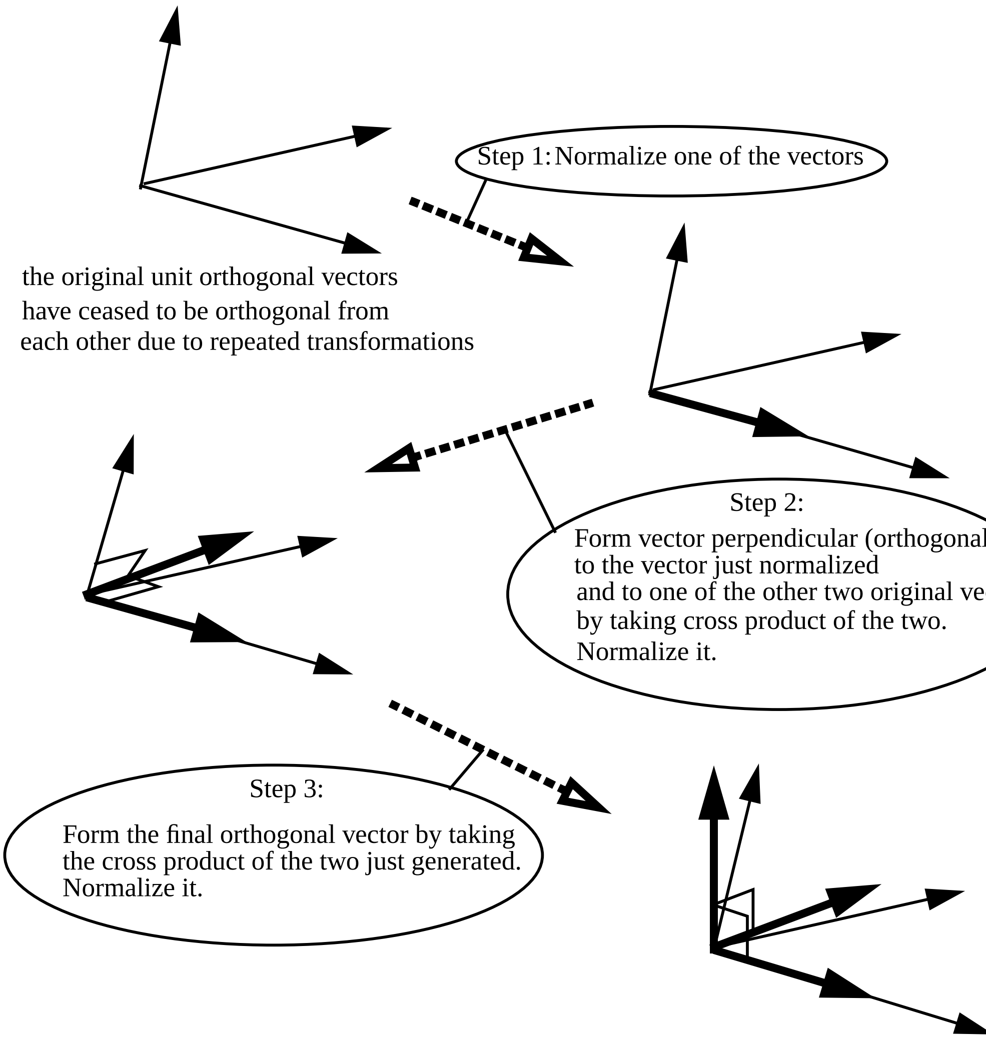


FIGURE 18. **Orthonormalization.**

.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

a) Positive 90 degree y-axis rotation

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

b) Negative 90 degree y-axis rotation

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

c) Half way between orientation representations

FIGURE 19. Direct interpolation of transformation matrix values can result in nonsense.

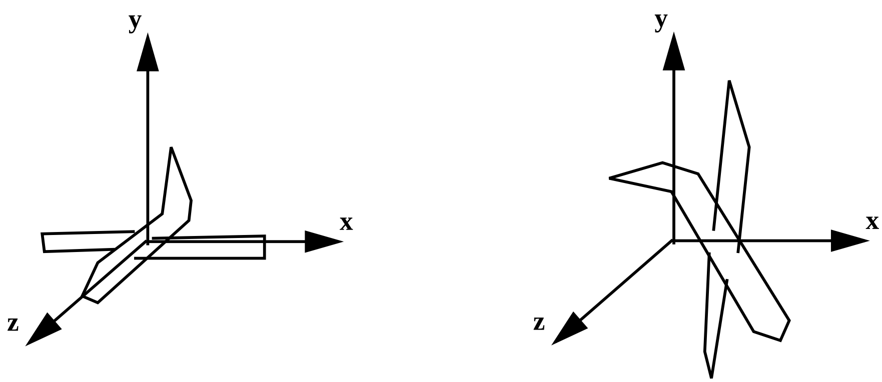


FIGURE 20. **Fixed angle representation.**

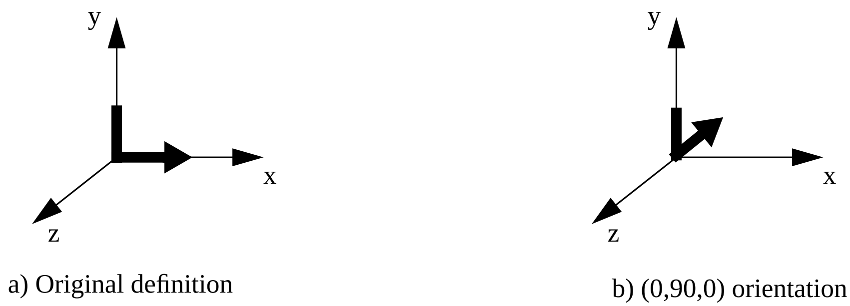
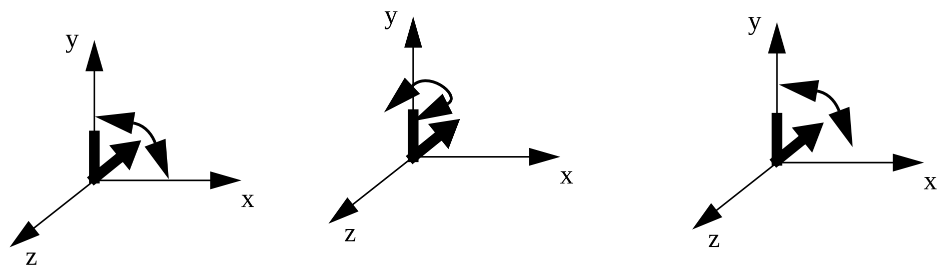


FIGURE 21. Fixed angle representation of (0,90,0).

.



a) (+/-ε,90,0) orientation b) (0,90+/-ε,0) orientation c) (0,90,+/-ε) orientation

FIGURE 22. **Effect of slightly altering values of fixed angle representation (0,90,0).**

(0,90,0) orientation

(90,45,90) orientation; the object lies in the y-z plane

FIGURE 23. Example orientations to interpolate.

.

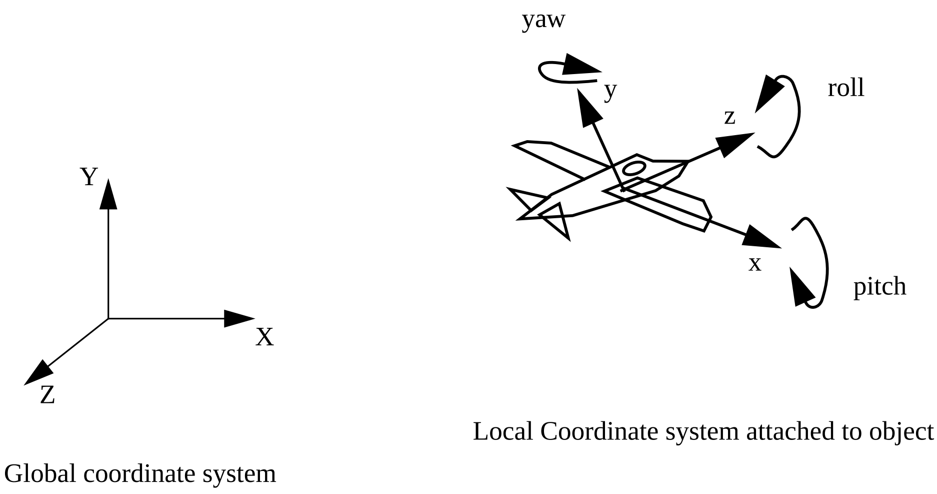


FIGURE 24. Euler angle representation.

$$R_y'(\beta)R_x(\alpha) = R_x(\alpha)R_y(\beta)R_x(\alpha)R_x(-\alpha) = R_x(\alpha)R_y(\beta) \quad \textbf{(EQ 18)}$$

$$R_z''(\gamma)R_y'(\beta)R_x(\alpha) = R_x(\alpha)R_y(\beta)R_z(\gamma)R_y(-\alpha)R_x(-\beta)R_y(\beta)R_x(\alpha) = R_x(\alpha)R_y(\beta)R_z(\gamma) \quad \textbf{(EQ 19)}$$

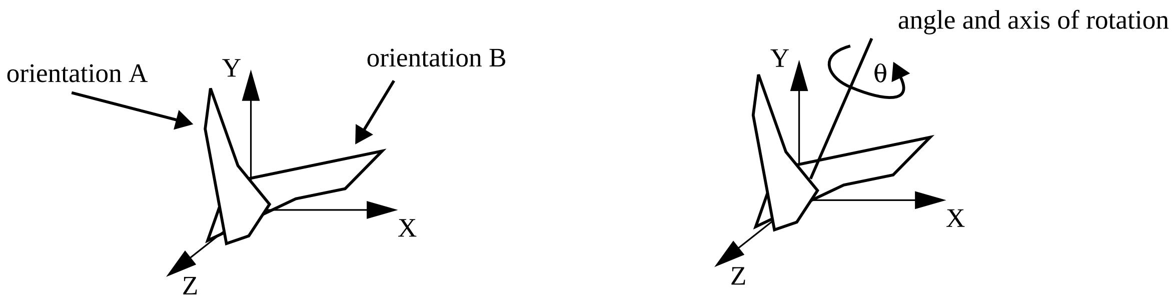
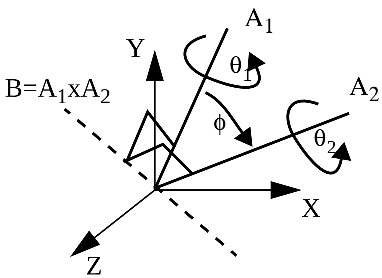


FIGURE 25. **Euler's Rotation Theorem implies that, for any two orientations of an object, one can be produced from the other by a single rotation about an arbitrary axis.**

!



$$B = A_1 \times A_2$$

$$\phi = \cos^{-1}\left(\frac{A_1 \bullet A_2}{|A_1||A_2|}\right)$$

$$A_k = R_B(k \cdot \phi)A_1$$

$$\theta_k = (1-k) \cdot \theta_1 + k \cdot \theta_2$$

FIGURE 26. Interpolating axis-angle representations.

.

$$[s_1, v_1] \cdot [s_2, v_2] = [s_1 \cdot s_2 - v_1 \bullet v_2, s_1 \cdot v_2 + s_2 \cdot v_1 + v_1 \times v_2] \quad \text{(EQ 20)}$$

$$[0, v_1] \cdot [0, v_2] = [0, v_1 \times v_2] \qquad \text{iff} \;\; v_1 \bullet v_2 = 0 \quad \text{(EQ 21)}$$

$$q^{-1} = (1/\|q\|)^2 \cdot [s, -v]$$
$$where \qquad \|q\| = \sqrt{s^2 + x^2 + y^2 + z^2} \quad \text{(EQ 22)}$$

$$q/\|q\| \quad q/(\|q\|) \quad \text{(EQ 23)}$$

$$v' = Rot(v) = q^{-1} \cdot v \cdot q \quad \text{(EQ 24)}$$

$$\begin{aligned} Rot_q(Rot_p(v)) &= q^{-1} \cdot (p^{-1} \cdot v \cdot p) \cdot q \\ &= ((pq)^{-1} \cdot v \cdot (pq)) \\ &= Rot_{pq}(v) \end{aligned} \quad \text{(EQ 25)}$$

$$Rot^{-1}(Rot(v)) = q \cdot (q^{-1} \cdot v \cdot q) \cdot q^{-1} = v \quad \text{(EQ 26)}$$

$$q = Rot_{\theta, (x, y, z)} = [\cos(\theta/2), \sin(\theta/2) \cdot (x, y, z)] \quad \text{(EQ 27)}$$

$$\begin{aligned} -q &= Rot_{-\theta, -(x, y, z)} \\ &= [\cos(-\theta/2), \sin((-\theta)/2) \cdot (-(x, y, z))] \\ &= [\cos(\theta/2), -\sin(\theta/2) \cdot (-(x, y, z))] \\ &= [\cos(\theta/2), \sin(\theta/2) \cdot x, y, z] \\ &= Rot_{\theta, (x, y, z)} \\ &= q \end{aligned} \quad \text{(EQ 28)}$$

27.