# Solving Systems of Linear Equations

# A System of Linear Equations

$n$ equations in $n$ unknowns $x_1$, $x_1$, …, $x_n$.

$$
\left\{
\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\
a_{i1}x_1 & + & a_{i2}x_2 & + & a_{i3}x_3 & + & \cdots & + & a_{in}x_n & = & b_i \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & a_{n3}x_3 & + & \cdots & + & a_{nn}x_n & = & b_n
\end{array}
\right.
$$

# In Matrix Form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

Or $Ax = b$

# Example

Equations:

$$\begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ 12x_1 - 8x_2 + 6x_3 + 10x_4 = 26 \\ 3x_1 - 13x_2 + 9x_3 + 3x_4 = -19 \\ -6x_1 + 4x_2 + x_3 - 18x_4 = -34 \end{cases}$$

In matrix form:

$$\begin{bmatrix} 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \\ 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -19 \\ -34 \end{bmatrix}$$

# Solving a System of Linear Equations

The system resulting from a sequence of the following operations is equivalent to the original system:

1. Multiply or divide an equation by a nonzero number.
2. Add an equation to another equation.
3. Substract an equation from another equation

# Example

$$\begin{cases} x_1 + 2x_2 = 5 \\ 2x_1 - 5x_2 = -8 \end{cases} \Leftrightarrow \begin{cases} -2x_1 - 4x_2 = -10 \\ 2x_1 - 5x_2 = -8 \end{cases} \Leftrightarrow \begin{cases} -2x_1 - 4x_2 = -10 \\ \quad - 9x_2 = -18 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 + 2x_2 = 5 \\ \quad - 9x_2 = -18 \end{cases}$$

Or

$$\begin{cases} x_1 + 2x_2 = 5 \\ 2x_1 - 5x_2 = -8 \end{cases} \Leftrightarrow \begin{cases} x_1 + 2x_2 = 5 \\ \quad - 9x_2 = -18 \end{cases}$$

# Gaussian Elimination: Example

$$\begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ 12x_1 - 8x_2 + 6x_3 + 10x_4 = 26 \\ 3x_1 - 13x_2 + 9x_3 + 3x_4 = -19 \\ -6x_1 + 4x_2 + x_3 - 18x_4 = -34 \end{cases}$$

$$\Leftrightarrow \begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ \quad - 4x_2 + 2x_3 + 2x_4 = -6 \\ \quad - 12x_2 + 8x_3 + x_4 = -27 \\ \quad + 2x_2 + 3x_3 - 14x_4 = -18 \end{cases}$$

$$\Leftrightarrow \begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ \quad\;\; - 4x_2 + 2x_3 + 2x_4 = -6 \\ \quad\quad\quad\quad\;\; + 2x_3 - 5x_4 = -9 \\ \quad\quad\quad\quad\;\; + 4x_3 - 13x_4 = -21 \end{cases}$$

$$\Leftrightarrow \begin{cases} 6x_1 - 2x_2 + 2x_3 + 4x_4 = 16 \\ \quad\;\; - 4x_2 + 2x_3 + 2x_4 = -6 \\ \quad\quad\quad\quad\;\; + 2x_3 - 5x_4 = -9 \\ \quad\quad\quad\quad\quad\quad\;\; - 3x_4 = -3 \end{cases}$$

- The system is now in upper triangular form.

# Gaussian Elimination

The method of Gaussian Elimination consists of two phases:

- Forward elimination: transforming the original system of linear equations into a system in upper triangular form.
- Backward substitution

# Backward Substitution

$$\begin{cases} 6x_1 & - & 2x_2 & + & 2x_3 & + & 4x_4 & = & 16 \\ & - & 4x_2 & + & 2x_3 & + & 2x_4 & = & -6 \\ & & & + & 2x_3 & - & 5x_4 & = & -9 \\ & & & & & - & 3x_4 & = & -3 \end{cases}$$

$$\Rightarrow \quad x_4 = 1, \quad x_3 = -2, \quad x_2 = 1, \quad x_1 = 3$$

# Gaussian Elimination: Algorithm

$$\begin{array}{l} \text{pivot} \rightarrow \\ \text{element} \\ \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} \begin{array}{l} \leftarrow \text{pivot row} \\ \\ \\ \\ \\ \\ \\ \end{array}$$

```
For $i \leftarrow 2$ to $n$ do        /* row $i$ $\leftarrow$ (row $i$) $-$ (row 1) $\times$ $(a_{i1} / a_{11})$ */
    $a_{ij} \leftarrow a_{ij} - a_{1j}\left(\frac{a_{i1}}{a_{11}}\right)$        $(1 \le j \le n)$
    $b_i \leftarrow b_i - b_1\left(\frac{a_{i1}}{a_{11}}\right)$
```

pivot →
element

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

← pivot row

```
For $i \leftarrow 3$ to $n$ do    /* row $i$ $\leftarrow$ (row $i$) $-$ (row 2) $\times$ ($a_{i2} / a_{22}$) */
    $a_{ij} \leftarrow a_{ij} - a_{2j}\left(\frac{a_{i2}}{a_{22}}\right) \qquad (2 \le j \le n)$
    $b_i \leftarrow b_i - b_2\left(\frac{a_{i2}}{a_{22}}\right)$
```

# Gaussian Elimination: Algorithm

```
For $k \leftarrow 1$ to $n-1$ do
    For $i \leftarrow k+1$ to $n$ do
        For $j \leftarrow k$ to $n$ do
            $a_{ij} \leftarrow a_{ij} - a_{kj}\left(a_{ik}/a_{kk}\right)$
            $b_i \leftarrow b_i - b_k\left(a_{ik}/a_{kk}\right)$
```

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & a_{ii} & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Remark: for efficiency, the multipliers $\left(a_{ik}/a_{kk}\right)$ should be computed before the $j$-loop.

# Backward Substitution

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & a_{ii} & & a_{in} \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

For $i \leftarrow n$ to 1 do

$$x_i \leftarrow \frac{1}{a_{ii}} \left( b_i - \sum_{j=i+1}^{n} a_{ij} x_j \right) \qquad \left( \text{Note: } a_{ii} x_i + a_{i,i+1} x_{i+1} \cdots + a_{in} x_n = b_i \right)$$

# Remarks

- The algorithm we have described is the basic (naive) version of Gaussian Elimination.
- Problems:
  - It will fail if some pivot element $a_{kk} = 0$.
  - Even if $a_{kk}$ is not zero but relatively small in magnitude, the computed solution may have considerable round-off errors.

# Example

$$\begin{cases} \varepsilon x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases} \Rightarrow \begin{cases} \varepsilon x_1 + x_2 = 1 \\ \left(1-\varepsilon^{-1}\right) x_2 = 2-\varepsilon^{-1} \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = \left(2-\varepsilon^{-1}\right) / \left(1-\varepsilon^{-1}\right) \\ x_1 = (1-x_2)/\varepsilon \end{cases}$$

On a single-precision machine, suppose $\varepsilon = 2^{-25}$.

$$\left.\begin{array}{l} \varepsilon^{-1} = \qquad\qquad 1.00\ldots0 \times 2^{25} \\ 2 = 1.0\ldots0 \times 2^{1} = 0.00\ldots01 \times 2^{25} \end{array}\right\} \Rightarrow 2-\varepsilon^{-1} = -\varepsilon^{-1}$$

Thus, $x_2$ will be computed as 1, and $x_1$ as 0.

# Now suppose we swap the two equations:

$$\begin{cases} x_1 + x_2 = 2 \\ \varepsilon x_1 + x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 + \qquad x_2 = 2 \\ \qquad (1-\varepsilon) x_2 = 1 - 2\varepsilon \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = \dfrac{1-2\varepsilon}{1-\varepsilon} \\ x_1 = 2 - x_2 \end{cases}$$

Again, suppose $\varepsilon = 2^{-25}$.

$$\left.\begin{aligned} 1 &= \qquad\qquad\qquad\quad 1.00\ldots0 \quad \times 2^0 \\ 2\varepsilon &= 1.0\ldots0 \times 2^{-24} = 0.00\ldots01 \times 2^0 \end{aligned}\right\} \Rightarrow 1 - 2\varepsilon = 1$$

Thus, $x_2$ will be computed as 1, and $x_1$ as 1.

# Testing solutions

$$\begin{cases} x_1 + x_2 = 2 \\ \varepsilon x_1 + x_2 = 1 \end{cases}$$

1. $x_1 = 0,\ x_2 = 1,\ \varepsilon = 2^{-25}$

2. $x_1 = 1,\ x_2 = 1,\ \varepsilon = 2^{-25}$ (correct)

# Partial Pivoting

- At step $k$, select the pivot row to be the one with the maximum absolute value among $a_{kk}, a_{k+1,k}, \ldots, a_{n,k}$.
- Then, swap.

$$\begin{bmatrix} a_{11} & \cdots & a_{1k} & \cdots & \cdots & a_{1n} \\ 0 & \ddots & \vdots & \cdots & & \vdots \\ 0 & 0 & a_{kk} & a_{k,k+1} & \cdots & a_{kn} \\ 0 & 0 & a_{k+1,k} & a_{k+1,k+1} & \cdots & a_{k+1,n} \\ 0 & 0 & a_{k+2,k} & a_{k+2,k+1} & \cdots & a_{k+2,n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{n,k} & a_{n,k+1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \\ x_{k+2} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \\ b_{k+1} \\ b_{k+2} \\ \vdots \\ b_n \end{bmatrix}$$

# Example: Partial Pivoting

- Swap the two rows since $1 > 2^{-25}$.

$$\begin{cases} 2^{-25}x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 = 2 \\ 2^{-25}x_1 + x_2 = 1 \end{cases}$$

# Complete Pivoting

- At step $k$, select the pivot row to be the one with the maximum absolute value among $a_{kk}, \ldots, a_{nn}$. Swap.
- More complicated.

$$\begin{bmatrix} a_{11} & \cdots & a_{1k} & \cdots & \cdots & a_{1n} \\ 0 & \ddots & \vdots & \cdots & & \vdots \\ 0 & 0 & a_{kk} & a_{k,k+1} & \cdots & a_{kn} \\ 0 & 0 & a_{k+1,k} & a_{k+1,k+1} & \cdots & a_{k+1,n} \\ 0 & 0 & a_{k+2,k} & a_{k+2,k+1} & \cdots & a_{k+2,n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & a_{n,k} & a_{n,k+1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1} \\ x_{k+2} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \\ b_{k+1} \\ b_{k+2} \\ \vdots \\ b_n \end{bmatrix}$$

# Example: Partial Pivoting

- Swap the two rows since $1 > 2^{-25}$.

$$\begin{cases} 2^{-25}x_1 + x_2 = 1 \\ x_1 + x_2 = 2 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 = 2 \\ 2^{-25}x_1 + x_2 = 1 \end{cases}$$

- How about this system?

$$\begin{cases} 2x_1 + 2^{26}x_2 = 2^{26} \\ x_1 + x_2 = 2 \end{cases}$$

# Example: Scaled Partial Pivoting

- Scale vector: $s^T = [13,\ 18,\ 6,\ 12]$. These values are scale factors.
- Scaled partial pivoting: pick the row with maximum $\dfrac{|a_{i1}|}{s_i}$.

$\left\{ \frac{3}{13}, \frac{6}{18}, \frac{6}{6}, \frac{12}{12} \right\}$. Row 3 is picked as the pivoting row.

$$\begin{pmatrix} 13 \\ 18 \\ 6 \\ 12 \end{pmatrix} \qquad \begin{bmatrix} 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -19 \\ -34 \\ 16 \\ 26 \end{bmatrix}$$

- Swap row 1 with row 3.

$$\begin{pmatrix} 6 \\ 18 \\ 13 \\ 12 \end{pmatrix} \qquad \begin{bmatrix} 6 & -2 & 2 & 4 \\ -6 & 4 & 1 & -18 \\ 3 & -13 & 9 & 3 \\ 12 & -8 & 6 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -34 \\ -19 \\ 26 \end{bmatrix}$$

- Pivot on row 1.

$$\begin{pmatrix} 6 \\ 18 \\ 13 \\ 12 \end{pmatrix} \qquad \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & 2 & 3 & -14 \\ 0 & -12 & 8 & 1 \\ 0 & -4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -18 \\ -27 \\ -6 \end{bmatrix}$$

- Pick the row (row 3) with maximum $\dfrac{|a_{i2}|}{s_i}$. Swap rows 2, 3.

$$\begin{pmatrix} 6 \\ 13 \\ 18 \\ 12 \end{pmatrix} \qquad \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \\ 0 & -4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -27 \\ -18 \\ -6 \end{bmatrix}$$

- Pivot on row 2.

$$\begin{pmatrix} 6 \\ 13 \\ 18 \\ 12 \end{pmatrix} \qquad \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -12 & 8 & 1 \\ 0 & 0 & 13/3 & -83/6 \\ 0 & 0 & -2/3 & 5/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -27 \\ -45/2 \\ 3 \end{bmatrix}$$

- Pivot on row 3.

$$\begin{pmatrix} 6 \\ 13 \\ 18 \\ 12 \end{pmatrix} \qquad \begin{bmatrix} 6 & -2 & 2 & 4 \\ 0 & -12 & 8 & 1 \\ 0 & 0 & 13/3 & -83/6 \\ 0 & 0 & 0 & -6/13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 16 \\ -27 \\ -45/2 \\ -6/13 \end{bmatrix}$$

- Back substitution: $x_4 = 1$

$$x_3 = \frac{-45/2 + 83/6 \cdot x_4}{13/3} = -2$$

$$x_2 = \frac{-27 - 8x_3 - x_4}{-12} = 1$$

$$x_1 = \frac{16 + 2x_2 - 2x_3 - 4x_4}{6} = 3$$

- Scale vector: $s = [s_1,\ s_2,\ \ldots,\ s_n]$, where $s_i = \max_{1 \le j \le n} |a_{ij}|$.
  - $s_i$ is the scale factor of row $i$.
  - When swapping two rows, you swap the corresponding scale factors, too.

$$\begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_i \\ \vdots \\ s_n \end{pmatrix} \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix}$$

# Algorithm: Gaussian Elimination with Scaled Partial Pivoting

```
For $k \leftarrow 1$ to $n-1$ do
    $i \leftarrow$ row in range $[k, n]$ with maximum $\frac{|a_{ik}|}{s_i}$
    Swap row $i$ with row $k$
    For $i \leftarrow k+1$ to $n$ do
        For $j \leftarrow k$ to $n$ do
            $a_{ij} \leftarrow a_{ij} - a_{kj}\left(a_{ik}/a_{kk}\right)$
        $b_i \leftarrow b_i - b_k\left(a_{ik}/a_{kk}\right)$
```

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & a_{ii} & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

# Without Physically Swapping Rows

- Physical rows: fixed
- Logical rows: rows in Gaussian elimination
- $l^T = [l_1,\ l_2,\ \ldots,\ l_n]$
  - $l_i$ indicates which physical row is logical row $i$
  - Initially, $l_i \leftarrow i$

# Example: Scaled Partial Pivoting w/o Swapping

- Scale vector: $s^T = [13,\ 18,\ 6,\ 12].$
- Index vector: $l^T = [1,\ 2,\ 3,\ 4].$
- Scaled partial pivoting: pick the row with maximum $\dfrac{|a_{l_i 1}|}{s_{l_i}}$.

$$\left\{ \frac{3}{13}, \frac{6}{18}, \frac{6}{6}, \frac{12}{12} \right\}.$$ Row 3 is picked as the pivoting row.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 13 \\ 18 \\ 6 \\ 12 \end{pmatrix} \quad \begin{bmatrix} 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -19 \\ -34 \\ 16 \\ 26 \end{bmatrix}$$

- Logically swap row 1 with row 3.

$$\begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 13 \\ 18 \\ 6 \\ 12 \end{pmatrix} \quad \begin{bmatrix} 3 & -13 & 9 & 3 \\ -6 & 4 & 1 & -18 \\ 6 & -2 & 2 & 4 \\ 12 & -8 & 6 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -19 \\ -34 \\ 16 \\ 26 \end{bmatrix}$$

- Pivot on logical row 1 (physical row 3).

$$\begin{pmatrix} 3 \\ 2 \\ 1 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 13 \\ 18 \\ 6 \\ 12 \end{pmatrix} \quad \begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \\ 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ -18 \\ 16 \\ -6 \end{bmatrix}$$

- Pick the row (logical row 3) with maximum $\dfrac{|a_{l_i 2}|}{s_{l_i}}$.

Swap logical rows 2, 3.

$$\begin{pmatrix} 3 \\ 1 \\ 2 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 13 \\ 18 \\ 6 \\ 12 \end{pmatrix} \quad \begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 2 & 3 & -14 \\ 6 & -2 & 2 & 4 \\ 0 & -4 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ -18 \\ 16 \\ -6 \end{bmatrix}$$

- Pivot on logical row 2.

$$\begin{pmatrix} 3 \\ 1 \\ 2 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 13 \\ 18 \\ 6 \\ 12 \end{pmatrix} \quad \begin{bmatrix} 0 & -12 & 8 & 1 \\ 0 & 0 & 13/3 & -83/6 \\ 6 & -2 & 2 & 4 \\ 0 & 0 & -2/3 & 5/3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -27 \\ -45/2 \\ 16 \\ 3 \end{bmatrix}$$

- Pivot on logical row 3.

$$\begin{pmatrix}3\\1\\2\\4\end{pmatrix} \quad \begin{pmatrix}13\\18\\6\\12\end{pmatrix} \quad \begin{bmatrix}0 & -12 & 8 & 1\\ 0 & 0 & 13/3 & -83/6\\ 6 & -2 & 2 & 4\\ 0 & 0 & 0 & -6/13\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-27\\ -45/2\\ 16\\ -6/13\end{bmatrix}$$

- Back substitution: $x_4 = 1$

$$x_3 = \frac{-45/2 + 83/6 \cdot x_4}{13/3} = -2$$

$$x_2 = \frac{-27 - 8x_3 - x_4}{-12} = 1$$

$$x_1 = \frac{16 + 2x_2 - 2x_3 - 4x_4}{6} = 3$$

# How to Modify the Algorithm?

```
For $k \leftarrow 1$ to $n-1$ do
    $i \leftarrow$ row in range $[k, n]$ with maximum $\dfrac{|a_{ik}|}{s_i}$
    Swap row $i$ with row $k$
    For $i \leftarrow k+1$ to $n$ do
        For $j \leftarrow k$ to $n$ do
            $a_{ij} \leftarrow a_{ij} - a_{kj}\left(a_{ik}/a_{kk}\right)$
        $b_i \leftarrow b_i - b_k\left(a_{ik}/a_{kk}\right)$
```

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & a_{ii} & a_{in} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$