Monte Carlo Integration

$$\int_a^b f(x)dx = ?$$

Monte-Carlo Integration

- $\int_a^b f(x)dx \approx (b-a)\frac{1}{n}\sum_{i=1}^{n} f(x_i)$

  where $x_1, x_2, \ldots, x_n$ are uniformly distributed random numbers in $[a,b]$.

- The error is $O\left(1/\sqrt{n}\right)$.

Monte-Carlo Integration

- $\int_{a_1}^{b_1}\int_{a_2}^{b_2}\int_{a_3}^{b_3} f(x, y, x)\; dx\; dy\; dz$

$$\approx (b_1 - a_1)(b_2 - a_2)(b_3 - a_3)\frac{1}{n}\sum_{i=1}^{n} f(x_i, y_i, z_i)$$

where $x_1,\ x_2,\ \ldots,\ x_n$ are random numbers in $[a_1,\ b_1]$

$y_1,\ y_2,\ \ldots,\ y_n$ are random numbers in $[a_2,\ b_2]$,

$z_1,\ z_2,\ \ldots,\ z_n$ are random numbers in $[a_3,\ b_3]$.

Monte-Carlo Integration

- In general,

$$\iint_{\Omega} f(x, y)\, dx\, dy$$

$$\approx \text{area}(\Omega) \cdot \left(\text{average of } f \text{ over } n \text{ random points in } \Omega\right)$$

$$\iiint_{\Omega} f(x, y, z)\, dx\, dy\, dz$$

$$\approx \text{volumn}(\Omega) \cdot \left(\text{average of } f \text{ over } n \text{ random points in } \Omega\right)$$

Example

$$\iint_{\Omega} f(x,y)\,dx\,dy$$

where $f(x,y) = \sin\sqrt{\ln(x+y+1)}$

$$\Omega = \left\{(x,y) : \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 \le \frac{1}{4}\right\}$$

$$\iint_{\Omega} f(x,y)\, dx\, dy$$

$$\approx \text{area}(\Omega) \cdot \left(\text{average height of } f \text{ over } n \text{ random points}\right)$$

$$= \pi r^2 \left[\frac{1}{n} \sum_{i=1}^{n} f(x_i, y_i) \right]$$

$$= \frac{\pi}{4n} \sum_{i=1}^{n} f(x_i, y_i)$$

Print intermediate estimates when $n = 1000,\ 2000,\ 3000,\ \ldots$

Example: Computing Volumns

What is the volumn of the following 3D space?

$$\Omega: \begin{cases} 0 \le x \le 1 \quad 0 \le y \le 1 \quad 0 \le z \le 1 \\ x^2 + \sin y \le z \\ x - z + e^y \le 1 \end{cases}$$

Define $f(x, y, x) = \begin{cases} 1 & \text{if } (x, y, x) \in \Omega \\ 0 & \text{otherwise} \end{cases}$

Then, $\text{volumn}(\Omega) = \int_0^1 \int_0^1 \int_0^1 f(x, y, x)\, dx\, dy\, dz$

$$
\begin{aligned}
\text{volumn}(\Omega) &= \int_0^1\int_0^1\int_0^1 f(x,y,x)\,dx\,dy\,dz \\
&\approx \frac{1}{n}\sum_{i=1}^{n} f(x_i, y_i, z_i) \\
&= \frac{m}{n}
\end{aligned}
$$

where m is the number of the n randomly generated points that are in Ω.

Example: Computing Volumns

- A cone: $z^2 = x^2 + y^2$
- A sphere: $x^2 + y^2 + (z-1) = 1$
- What is the volumn of the space above the cone and inside the sphere?
- The sphere is contained in a $2 \times 2 \times 2$ cube.
- Algorithm:

  1. Generate n random points in the cube.

  2. Count how many of these points are in Ω; say, m.

  3. volumn(Ω) $\approx 8 \times \dfrac{m}{n}$.