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Fast Reconstruction of Curves with Sharp Corners

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ABSTRACT

We present an algorithm to reconstruct a collection of piecewise smooth simple closed curves in the plane from a set of n sample points in $O(n \log n)$ time. We prove our algorithm correctly reconstructs the curves assuming certain sampling conditions which are based on the minimum angle made by tangents at any corner point but does not include any assumptions about the uniformity of the sampling.

1. Introduction

Let Γ be a collection of simple closed curves in the plane. Let P be a finite set of points from Γ, called the *sample set* of Γ. Our objective is to reconstruct Γ from the sample set P. More specifically, two sample points $p, p' \in P$ are *adjacent* (on Γ) if an interval on Γ from p to p' does not contain any other points of P. We wish to form polygonal curves connecting all pairs of adjacent sample points on Γ.

Numerous methods have been proposed which guarantee reconstruction of smooth closed curves under uniform sampling conditions including α-shapes[5,12], β-skeletons[16], r-regular shapes[4] and minimum spanning trees[6]. Edelsbrunner gives a survey of these techniques in [11].

In 1998, Amenta, Bern and Eppstein[3] presented an algorithm, called CRUST, which was guaranteed to reconstruct a smooth curve from a set of sample points under appropriate sampling conditions. As opposed to previous algorithms, the sampling was not required to be uniform. Instead, the sampling density could vary over the curve, increasing in "crowded" areas or in areas of high curvature and decreasing in flat, isolated areas of the curve.

The CRUST algorithm required two separate Voronoi diagram computations. Dey and Kumar[9] and Gold[15] gave simplified versions of the algorithm which required only one such computation.

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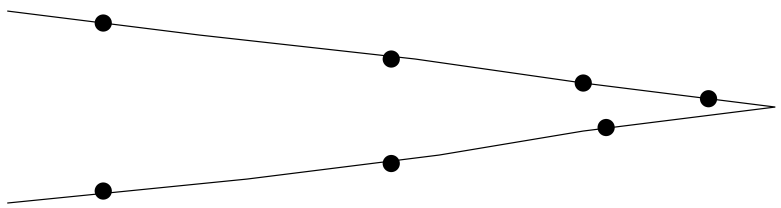


Figure 1: Nearest neighbors are not necessarily adjacent.

There were two major drawbacks to the CRUST algorithm and the variations. First, the algorithm reconstructed closed curves, but was unable to handle open curves or collections of open curves. Dey, Mehlhorn and Ramos[10] presented the CONSERVATIVE-CRUST algorithm which closely checked and eliminate some of the reconstruction edges that might have been added by CRUST. They gave certain reconstruction guarantees, even for open curves, under appropriate sampling conditions.

The second problem was that CRUST required the curves to be smooth to guarantee reconstruction. It often failed, both in theory and in practice, to produce correct reconstructions of curves with corner points. One problem is that in the neighborhood of a corner, two non-adjacent sample points can be much closer to each other than to any other points. (See Figure 1.) Increasing the required sampling density will not help, since the same pattern can occur at a smaller scale arbitrarily close to the corner.

In [14], Giesen showed that for sufficiently dense samplings, the travelling salesman tour will be a correct reconstruction of a piecewise smooth simple closed curve. The curves must satisfy the condition that the angle made by the tangents at any corner point must be non-zero. In general, the travelling salesman tour is difficult to compute. Althaus and Mehlhorn[1] showed how to compute it in polynomial time for curve reconstruction. Their solution is based on formulating the problem as a linear program and applying the Ellipsoid method. They do not give the exact polynomial which is dependent on the bit size in the cost function of the linear program. In practice, they report using the simplex method with cutting planes which seems to work well even though it takes potentially exponential time. Althaus et. al. in [2] report that the travelling salesman method can take up to thirteen times longer than CRUST and its variants on data sets of approximately three thousand sample points.

Althaus and Mehlhorn[1] also showed how to extend the travelling salesman approach to handle an open curve or a collection of closed curves. They were unable to apply the method to collections of open and closed curves.

In [8], we presented a heuristic for reconstructing collections of piecewise-smooth curves from a set of sample points. We used the sampling conditions of Amenta, Bern and Eppstein but modified those sampling conditions in the neighborhood of the corner points. Our algorithm did not guarantee a correct reconstruction on piecewise-smooth curves, but in practice did fairly well.

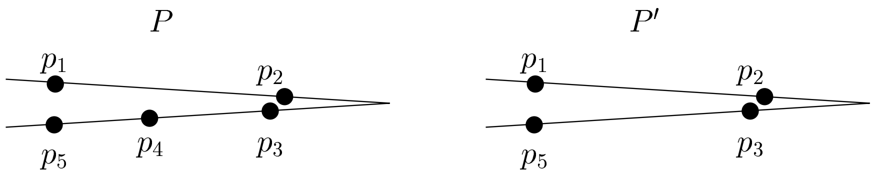


Figure 2: Point set P' satisfies the angular sampling condition of Ramos and Funke for $\theta_{angle} = 120°$ but point set $P \subseteq P'$ does not, since $\angle(p_1, p_4, p_2) \geq 120°$.

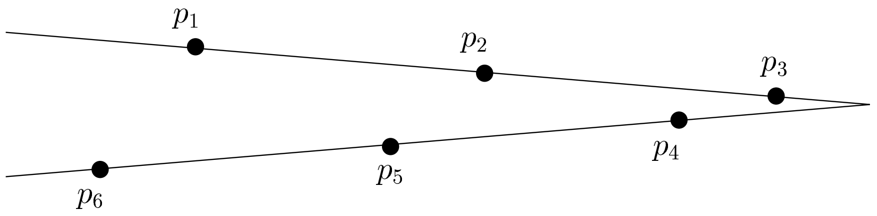


Figure 3: A uniform sampling which does not satisfy the angular sampling condition of Ramos and Funke for $\theta_{angle} = 120°$, since $\angle(p_2, p_4, p_3) \geq 120°$.

Recently, Ramos and Funke[13] presented an algorithm for reconstructing collections of open and closed piecewise-smooth curves. Their algorithm used local criteria to find reconstruction edges and estimate sharp corner points and endpoints. Their algorithm runs in $O(n_c n^2)$ time where n is the number of sample points and n_c is the number of sharp corners. In practice, they claim that their algorithm runs in $O(n_c n)$ time.

Ramos and Funke introduced a new sampling condition which was substantially different from those in previous papers. For any edge (p_1, p_2) of the correct reconstruction and any other sample point q, angle $\angle(p_1, q, p_2)$ was bounded from above by a constant θ_{angle}. Their reconstruction guarantees depend upon this sampling condition, particularly in the neighborhood of sharp corners. In practice, they recommend a value of 120° for this bound.

The angular sampling condition of Ramos and Funke has some odd consequences. For one thing, a subset of some sample set P may satisfy this sampling condition while P does not. (See Figure 2.) For another, an arbitrarily dense sampling may fail to satisfy this sampling condition near a corner angle less than $180 - \theta_{angle}$. (See again Figure 2, where P could be scaled down to be arbitrarily close to the corner.) Even a dense uniform sampling where the length along the curve between every two adjacent sample points is equal may not satisfy this condition near sharp corners. (See Figure 3.)

If θ_{angle} is chosen to be larger than $180 - \mu$, where μ is the minimum angle at any corner, then the angular sampling condition will be guaranteed in the neighborhood of corners. However, the angular sampling condition must also be satisfied in the smooth portions of the curve. Thus as θ_{angle} increases, the required sampling density in the smooth portions of the curve also increase.

In this paper, we present an algorithm for reconstructing a collection of piecewise-smooth closed curves and prove that our algorithm produces a correct reconstruction under appropriate sampling conditions. We require the same condition as Giesen that the angle made by the tangents at any corner point must be non-zero. Our algorithm is based on ideas from our algorithm Gathan in [8], but is carefully structured to guarantee a provably correct reconstruction.

The only parameter required by our algorithm is the minimum angle over all the corner points. This minimum angle affects the sampling condition for the three or four points nearest to each corner. It has no affect on any other portions of the curve.

Given the minimum angle μ over all the corner points, Ramos and Funke could apply their algorithm using a value $\theta_{angle} > 180 - \mu$. In this sense, their algorithm is more general than ours. On the other hand, Ramos and Funke's algorithm requires that the sampling density in smooth portions of the curve increases with θ_{angle}. The minimum angle μ affects our sampling density only in a small neighborhood of each corner. In this sense, our sampling condition is less restrictive than the one given by Ramos and Funke.

The algorithm by Ramos and Funke guarantees proper reconstruction in the presence of open curves under appropriate sampling conditions. Our algorithm does not. Our algorithm may also be affected by undersampling in critical portions of the curve and by noise from outliers. We present some simple modifications to our original algorithm which improve its performance under these various conditions. The modified algorithm still is guaranteed to run correctly on collections of properly sampled simple closed curves but no guarantees are made for open curves. Guaranteeing correct reconstruction of even smooth open curves requires an additional level of algorithmic and analytic complexity which is far beyond our simple modifications. (See [10].)

Both our original algorithm and the modified practical version run in $O(n \log n)$ time on a set of n sample points. The $n \log n$ time is required for computation of the Delaunay triangulation and for an initial sorting of the Delaunay edges. The rest of the algorithm runs in linear time.

In Section 2, we discuss the basic ideas behind our algorithm. Section 3 describes the algorithm itself. Section 4 discusses some properties of the neighborhoods of corners. Section 5 gives the sampling conditions required for guarantees of reconstruction. In Section 6, we prove that the algorithm will correctly reconstruct collections of simple closed curves under the appropriate sampling conditions. Section 7 analyzes the running time of the algorithm. Section 8 describes the practical modifications of the algorithm which help it handle endpoints, undersampling and outliers. Section 9 gives some experimental results. Finally, Section 10 concludes with some open problems.

2. Voronoi Diagrams of Sample Sets

Our algorithm belongs to a whole slew of curve reconstruction algorithms [3,4,9,10,15,17] which are based on the Voronoi diagrams and Delaunay triangulation of the sample point set. All the algorithms produce a reconstruction which is a subset of the Delaunay edges of the sample points. They differ in how they use the Voronoi diagram to determine which Delaunay edges to choose.

Amenta, Bern and Eppstein noted that if a sampling of a smooth curve is sufficiently dense, then the Voronoi cells are long and thin in a direction approximately parallel to the curve normals. A *pole* of a sample point p is the furthest point from p in its Voronoi cell. If the Voronoi cell is unbounded, then the pole is one of the points in the Voronoi cell at infinity. The *pole direction*, $d(p)$, is the direction from p to the pole. If a sampling curve is sufficiently dense, then the pole direction $d(p)$ is a good approximator for the curve normal up to orientation.

The basis for our heuristic in [8] and our algorithm in this paper is the observation that even if a curve contains sharp corners, the Voronoi cells are almost all still long and thin in a direction approximately parallel to the curve normals. Thus the pole directions are still good approximators for the curve normals (Lemma 9.) The only exceptions are the Voronoi cells of sample points which are located on or adjacent to the corners.

Pole directions are good approximators of the curve normal only up to orientation. Two adjacent sample points on the curve may have pole directions which are almost opposite. However, in a sufficiently small neighborhood of a sharp corner, all pole directions will point on the same side of the curve, "away" from the corner (Lemma 7.) We make extensive use of this fact to avoid non-reconstruction edges which cut across a corner.

Another component of our algorithm is the *ratio test* which compares the length of a Delaunay edge (p_1, p_2) to the distance from p_1 (or equivalently from p_2) to the endpoints of the dual Voronoi edge. (In [8], we used an equivalent ratio test which compared the distance from the midpoint of (p_1, p_2) to the endpoints of the dual Voronoi edges.) Let L be the distance from the p_1 to the farthest endpoint of the dual Voronoi edge on a given side. If there is no endpoint on that side, let L be 0. The Delaunay edge (p_1, p_2) passes the ratio test on the given side for ratio σ, if $L/|e|$ is greater than σ. We use the ratio test to avoid non-reconstruction edges which cut across a corner.

3. Algorithm

Our algorithm is based on the nearest neighbor reconstruction algorithm by Dey and Kumar[9]. Our algorithm starts by constructing the Delaunay triangulation and Voronoi diagram of the sample points. It then reconstructs the curve in four steps. First, it verifies that certain nearest neighbor edges are part of the reconstruction. These verified edges are "seeds" for the rest of the reconstruction. The second step extends the reconstruction along adjacent sample points whose poles point in approximately the same direction. This will reconstruct most of the curve in the neighborhood of each corner point, except perhaps for edges adjacent to the corner. The third step adds any of the missing edges adjacent to the corners. The last step adds any edges missed by the first three, essentially, any edges in smooth portions of the curve where the pole direction flips from one side of the curve to the other.

Sharp_Reconstruction(P, n, α)
/* P is a set of n sample points */
/* α is a strict lower bound on the minimum corner angle */
1. Construct the Delaunay triangulation, $\mathcal{T}$, of P;
2. Connect_Nearest_Neighbors($\mathcal{T}$); /* Connect nearest neighbors */
3. Extend_Pole_Pole($\mathcal{T}$, α); /* Extend using pole-pole and angle tests */
4. Add_Corner_Edges($\mathcal{T}$, α); /* Add edges adjacent to corners */
5. Extend_Smooth($\mathcal{T}$); /* Extend to smooth portions of the curve */

Figure 4: Curve reconstruction algorithm.

Connect_Nearest_Neighbors($\mathcal{T}$)
/* Connect nearest neighbors which pass the pole-pole test */
1. **for** each vertex p_1 of $\mathcal{T}$ **do**
2. $\quad$ $p_2 \leftarrow$ Nearest-Neighbor(p_1);
3. $\quad$ **if** $\angle(d(p_1), d(p_2)) \leq 60°$, **then**
4. $\quad\quad$ Mark (p_1, p_2) as a reconstruction edge;
5. $\quad$ **endif**
6. **endfor**

Figure 5: Connect nearest neighbors algorithm.

(See Figure 4.)

In the neighborhood of a corner, nearest neighbors are not necessarily adjacent reconstruction vertices (See Figure 1.) However, if nearest neighbors are not adjacent, then their pole directions will be vastly different (Lemma 12.) Thus in the first reconstruction step, we check the angle between the pole directions at the endpoints of an edge. If the angle is large, we do not add the edge to the reconstruction at this time. (See Figure 5.)

The second step is the most difficult. The heart of the second step is a set of criteria which permits us to eliminate some edges from consideration.

Consider a vertex p_1 which has exactly one edge (p_0, p_1) in the current recon-

Vor_Del_Ratio($\mathcal{T}$, p_0, p_1, p_2)
/* return Voronoi-Delaunay ratio for Delaunay edge (p_1, p_2) */
1. If $d(p_1)$ is on the left of curve (p_0, p_1, p_2), then
2. $\quad$ $VRad \leftarrow$ radius of largest empty circle through p_1 and p_2 whose center lies on left side of (p_1, p_2). (0 if there is no such circle;)
3. else
4. $\quad$ $VRad \leftarrow$ radius of largest empty circle through p_1 and p_2 whose center lies on right side of (p_1, p_2). (0 if there is no such circle;)
5. return($VRad/|(p_1, p_2)|$);

Figure 6: Compute Voronoi-Delaunay ratio.

Shortest_Potential($\mathcal{T}$, p, α)
/* return shortest potential edge incident on p */
1. $(q_0, q_1) \leftarrow$ shortest edge incident on p such that for each endpoint q_i of e:
    - At most one edge incident on q_i is marked as a reconstruction edge;
    - If q_i is incident on a reconstruction edge (q_i, q), and $\angle(q_{1-i}, q_i, q) \leq 30°$, then $\angle(d(q_{1-i}), d(q)) \geq 90°$;
      /* Note q_{1-i} is the other endpoint of e */
    - If q_i is incident on a reconstruction edge (q_i, q) and $\angle(q_{1-i}, q_i, q) \leq 135°$, then Vor_Del_Ratio($\mathcal{T}$, q, q_i, q_{1-i}) $\geq \frac{1}{2\sin(\alpha)}$;
2. **return**((q_0, q_1));

Figure 7: Criteria for shortest potential edges.

Verify_Edge($\mathcal{T}$, p_0, p_1, p_2)
/* verify that (p_1, p_2) is a reconstruction edge */
/* edge (p_0, p_1) is an identified reconstruction edge */
1. **if** $\angle(p_0, p_1, p_2) \geq 120°$ **then**
2.   **if** $\angle(d(p_1), d(p_2)) \leq 60°$ **then**
3.   **return** (**true**); /* verified reconstruction edge */
4. $\mathbf{n} \leftarrow$ normal to (p_1, p_2) on same side of curve (p_0, p_1, p_2) as $d(p_1)$;
5.   **if** $(\angle(d(p_1), d(p_2)) \leq 120°)$ **and** $(\angle(\mathbf{n}, d(p_2)) \leq 120°)$ **then**
6.   **if** Vor_Del_Ratio($\mathcal{T}$, p_0, p_1, p_2) $\geq \frac{1}{2\sin(\alpha)}$ **then**
7.   **return** (**true**); /* verified reconstruction edge */
8. **return** (**false**); /* unable to verify reconstruction edge */

Figure 8: Verify that edge (p_1, p_2) is a reconstruction edge.

Extend_Pole_Pole($\mathcal{T}$, α)
/* Add shortest potential edges passing pole-pole and angle tests */
1. Add all vertices to stack S;
2. **while** $S \neq \emptyset$ **do**
3. $p_1 \leftarrow S.Pop()$;
4.   **if** p_1 is incident on exactly one reconstruction edge (p_0, p_1) **then**
5. $(p_1, p_2) \leftarrow$ Shortest_Potential($\mathcal{T}$, p_1, α);
6.   **if** Verify_Edge($\mathcal{T}$, p_0, p_1, p_2) **then**
7.   Mark (p_1, p_2) as a reconstruction edge;
8.   Add p_2 and all neighbors of p_1 and p_2 to stack S;
9. **endwhile**

Figure 9: Extend using pole-pole and angle tests.

struction. (We use (p_0, p_1) to denote both the line segment connecting points p_0 and p_1 and the Delaunay edge connecting vertices p_0 and p_1.) We would like to find the missing edge. Possible candidates include the shortest edge (p_1, p_2) incident on p_1 (other than (p_0, p_1)) or the shortest edge (p_1, p_2) which makes an angle of at least ninety degrees with (p_0, p_1). Neither of these quite suffice. Instead, we apply three criteria to eliminate edges incident on p_1 from consideration.

Let (p_0, p_1) be an edge of the Delaunay triangulation and (p_1, p_2) be a reconstruction edge. We first note that if we have already found two reconstruction edges incident on p_0 or p_1, then (p_0, p_1) is not a reconstruction edge. Secondly, if $\angle(p_0, p_1, p_2)$ is at most 30 degrees and the angle between the pole directions $d(p_0)$ and $d(p_2)$ is small, then (p_0, p_1) is not a reconstruction edge. Finally, the pole direction $d(p_1)$ points to one side of the curve (p_0, p_1, p_2). If edge (p_0, p_1) fails the ratio test on this side and $\angle(p_0, p_1, p_2)$ is at most 135 degrees, then (p_0, p_1) is not a reconstruction edge. We use the ratio $\frac{1}{2\sin(\alpha)}$ for the ratio test where α is a strict lower bound on the minimum corner angle. (See Figures 6 and 7.)

The justification for the first criterion is obvious. For the second, if (p_0, p_1) and (p_1, p_2) were correct reconstruction edges and $\angle(p_0, p_1, p_2)$ was less than 30 degrees, then p_1 would be adjacent to a corner with angle approximately 30 degrees or less. Since the pole directions all point away from the corner, the pole directions at p_0 and p_2 should point away from one another (Lemma 11.) If they do not, then (p_1, p_2) is not a reconstruction edge.

For the third criterion, if (p_0, p_1) and (p_1, p_2) were correct reconstruction edges and $\angle(p_0, p_1, p_2)$ was at most 135 degrees, then again p_1 would be adjacent to a corner (Lemma 3.) Our sampling condition in the neighborhood of a corner will guarantee that (p_1, p_2) satisfies the ratio condition on its outer side which is the side pointed to by the pole $d(p_1)$ of p_1 (Lemmas 7 and 16.)

An edge which satisfies all three criterion is called a *potential* edge. For each vertex p_1 which has exactly one edge (p_0, p_1) in the current reconstruction, we consider the shortest potential edge (p_1, p_2) incident on p_1. Unfortunately, a non-reconstruction edge which cuts across a corner may be the shortest potential edge incident on p_1. We apply a number of tests to verify that the shortest potential edge is truly a reconstruction edge. (See Figure 8.)

First, we require that the angle between (p_1, p_2) and the reconstruction edge (p_0, p_1) be large (greater than $120°$.) This requirement stops us from accidentally creating sharp corners in smooth parts of the curve. Next, we require that angle between $d(p_1)$ and $d(p_2)$ be small (less than $60°$.) Since the pole directions between the endpoints of edges which cross a corner vary greatly, non-reconstruction edges will not pass this test.

Almost all reconstruction edges (p_1, p_2) satisfy the condition that $\angle(d(p_1), d(p_2)) \leq 60°$. However, if p_2 is adjacent to a corner g, then (p_1, p_2) may not satisfy this condition. We loosen this condition by replacing the bound of $60°$ with a bound of $120°$ on the pole angle but augment with a ratio test and a bound on the angle between $d(p_2)$ and the normal $\mathbf{n}$ to (p_1, p_2). We prove that non-reconstruction edges which cross a corner fail these tests.

```
Add_Corner_Edges(𝒯, α)
/* Add edges adjacent to a corner */
Add all vertices to stack S;
while S ≠ ∅ do
        p1 ← S.Pop();
        if p1 is incident on exactly one reconstruction edge (p0, p1) then
                (p1, p2) ← Shortest_Potential(𝒯, p1, α);
                if ∠(p0, p1, p2) ≥ 70° then
                        Mark (p1, p2) as a reconstruction edge;
                        Add p2 and all neighbors of p1 and p2 to stack S;
                endif
        endif
endwhile
```

Figure 10: Add edges adjacent to the corner.

The shortest potential edge incident on p_1 may not be a reconstruction edge. However, as more edges are added to the reconstruction, the set of potential edges shrinks and the shortest potential edge incident on p_1 will change. We prove that eventually the shortest potential edge incident on p_1 will be a correct reconstruction edge.

The second reconstruction step may fail to add one or two edges at each corner, either because such edges make an angle less than 120 degrees with the other reconstruction edges or because the pole directions at vertices adjacent to the corner vary greatly. In the third reconstruction step, we again consider the shortest potential edge (p_1, p_2) incident on p_1. However, we remove the conditions that the pole directions at p_1 and p_2 point in the same direction. If $\angle(p_0, p_1, p_2)$ is at least seventy degrees, then we add (p_1, p_2) to the set of reconstruction edges. (See Figure 10.)

The first three steps complete the reconstruction in the neighborhoods of sharp corners. However, there may still be missing edges in the smooth parts of the curve or in the neighborhood of angles which are not very sharp. The last step adds uses Dey and Kumar's algorithm to add in these edges. We extend the reconstruction from an edge (p_0, p_1) by adding the shortest potential edge making an angle of at least 90 degrees (p_0, p_1). (See Figure 11.)

4. Neighborhoods of Corners

Let $G(\Gamma)$ denote the set of non-smooth "corner" points of Γ. The *medial axis* of a curve Γ is the set of points in the plane which have more than one closest point on Γ. A curve Γ splits a sufficiently small disk, $\mathbb{B}_g$, around a corner point $g \in G(\Gamma)$ into two parts. One of these parts contains the medial axis which touches the curve at the corner. This part is called the *inner side* of Γ (in the neighborhood of g) while the other part is the *outer side.* (See Figure 12.) A major observation (Lemma 7) is that in the neighborhood of a corner the pole directions always point to the outer

```
Extend_Smooth(T)
/* Add remaining edges in smooth portions of the curve */
for each vertex p1 of T do
        p2 ← Nearest-Neighbor(p1);
        if p1 and p2 are not incident on any reconstruction edges, then
                Mark (p1, p2) as a reconstruction edge;
        endif
endfor
Add all vertices to stack S;
while S ≠ ∅ do
        p1 ← S.Pop();
        if p1 is incident on exactly one reconstruction edge (p0, p1) then
                (p1, p2) ← shortest edge incident on p1 such that ∠(p0, p1, p2) ≥ 90°;
                Mark (p1, p2) as a reconstruction edge;
                Add p2 to stack S;
        endif
endwhile
```

Figure 11: Add remaining edges in smooth portions of the curve.

side of the curve.

We define the "outer side" of any line segment (p, p') connecting points $p, p' \in \Gamma \cap \mathbb{B}_g$ as follows. Without loss of generality, orient $\Gamma \cap \mathbb{B}_g$ so that the outer side is on the left. This orientation order all the points on $\Gamma \cap \mathbb{B}_g$. Let p, p' be two distinct points in $\Gamma \cap \mathbb{B}_g$, where p precedes p'. The *outer side* of line segment (p, p') is the left side of this line segment when directed from p to p'. (See Figure 12.) Note that this outer side is defined for any pair of points $p, p' \in \Gamma \cap \mathbb{B}_g(r_g)$, not just adjacent sample points.

We say that a sample point p is *adjacent* to a corner g, if g is a sample point and p equals g or if g is not a sample point and there are no sample points on the curve between p and g. A correct reconstruction edge (p, p') is *adjacent* to a corner g if either p or p' (or both) are adjacent to g.

The intersection of Γ and a suitably small disk around a corner point g is an open curve. The corner point g splits this curve into two parts or *legs*. The corner point g lies on both legs. A *cross edge* is a Delaunay edge which connects two points which do not lie on the same leg. If the corner point g is not a sample point, then the cross edge between the two sample points "adjacent" to g is a reconstruction edge. All other cross edges are not reconstruction edges.

For each corner point $g \in G(\Gamma)$, define the corner angle α_g as $\lim_{p,p' \to g} \angle(p, g, p')$ where p is on one leg of Γ and p' is on the other. If the curve makes a very sharp turn at g, then this angle is very small.

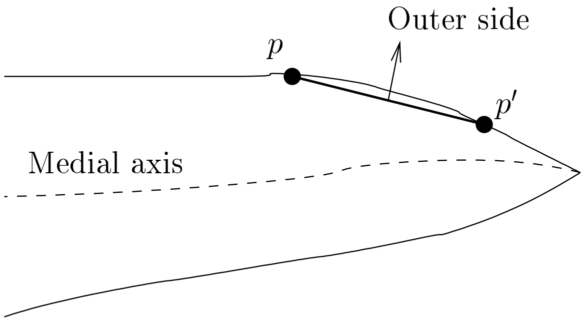


Figure 12: Outer side of a line segment in the neighborhood of a corner.

5. Sampling Conditions

If a curve is undersampled, then there is, of course, no way to reconstruct the original curve. We wish to give specific sampling conditions which will guarantee that the curve is correctly reconstructed. Moreover, these sampling conditions should vary over the curve, with more samples required in the neighborhood of small features and fewer required in the neighborhood of large ones.

The *medial axis* of a curve Γ is the set of points in the plane which have more than one closest point on Γ. Amenta, Bern and Eppstein defined the *local feature size*, $f(q)$, at a point $q \in \Gamma$ to be the distance from q to the medial axis of Γ. They required that the sample set satisfy the following sampling condition for every point on Γ:

Sampling condition $R_1(\epsilon)$: Sample set P satisfies sampling condition $R_1(\epsilon)$ at point $p \in \Gamma$ if every point on the closed intervals of Γ between p and the two sample points adjacent to p are at most distance $\epsilon f(p)$ from p.

This sampling condition is a modification of the one given by Amenta, Bern and Eppstein but it is essentially equivalent. (See Dey and Kumar [9] for theorems relating Amenta, Bern and Eppstein's original condition and the distance between adjacent sample points.) Note that if P satisfies sampling condition $R_1(\epsilon f(p))$ at a sample point p, then the two reconstruction edges incident on p have length at most $\epsilon f(p)$. Amenta, Bern and Eppstein proved that for suitably small ϵ, the CRUST algorithm correctly reconstructed smooth curves.

Unfortunately, the sampling condition based on local feature size is not appropriate for curves with corners. The medial axis touches these curves at the corners and so the local feature size goes to zero as points approach the corners. Thus the local feature size sampling condition would require using an infinite number of points in the neighborhood of corners. Thus in the neighborhood of a corner $g \in \Gamma$, we drop sampling condition $R_1(\epsilon)$. Instead we use the following simple sampling condition, which guarantees that the distance between sample points is not too large:

Sampling condition $R_2(\delta)$: Sample set P satisfies sampling condition $R_2(\delta)$ at point $p \in \Gamma$ if every point on the two closed intervals of Γ between p and the two sample points adjacent to p are at most distance δ from p.

Note that if P satisfies sampling condition $R_2(\delta)$ at a sample point p, then the two reconstruction edges incident on p have length at most δ. Sampling condition $R_1(\epsilon)$ is the same as sampling condition $R_2(\epsilon f(p))$ where $f(p)$ is the feature size at point p.

For each corner point $g \in G(\Gamma)$, we actually define two neighborhoods, one inside the other, and use one sample distance, δ_g, in the larger neighborhood, and a smaller sample distance, $\hat{\delta}_g \leq \delta_g$, in the smaller one. The value of δ_g depends on the size of the larger neighborhood of g and the corner angle, α_g. The value of $\hat{\delta}_g$ also depends on the size of the larger neighborhood of g. However, in place of a dependence on α_g, the value of $\hat{\delta}_g$ depends upon a constant α which is a strict less than the minimum angle at any corner point. By using $R_2(\hat{\delta}_g)$, we ensure that the edges with endpoints adjacent to corners pass a ratio test. Of course, we could have required the stricter sampling condition $R_2(\hat{\delta}_g)$ throughout the larger neighborhood, but it was not necessary for the correctness of our algorithm.

We first define the larger neighborhood for sampling condition $R_2(\delta_g)$. This neighborhood should be very far from any other corner points and should contain only the part of the medial axis which touches g. In this neighborhood, the two legs of Γ should be almost straight.

For any point $p \in \mathbb{R}^2$ and $r > 0$, let $\mathbb{B}_p(r)$ be the disk (ball) of radius r around p. If the disk around a corner point g intersects Γ in a connected curve with two endpoints, then g divides this curve into two legs. We need both upper and lower bounds on the angle made by points on the two legs in $\mathbb{B}_p(r)$. Let $\alpha_g^U(r)$ be the upper bound $\limsup \angle(p, g, p')$ where p and p' are points on different legs of the curve $\Gamma \cap \mathbb{B}_g(r)$. Let $\alpha_g^L(r)$ be the lower bound $\liminf \angle(p, p', p'')$ where p, p', p'' are three distinct points in the given order in $\Gamma \cap \mathbb{B}_g(r)$. Let N_g be the distance from g to the nearest corner point $g' \in G(\Gamma) - \{g\}$. Note that $\alpha_g^U(r)$ bounds the angle at the corner g whereas $\alpha_g^L(r)$ bounds the angle at every point p. Note also that $\alpha_g^L(r) \leq \alpha_g \leq \alpha_g^U(r)$ for every r.

For each corner point $g \in G(\Gamma)$, choose a ball $\mathbb{B}_g(r_g)$ around g such that:

1. $\Gamma \cap \mathbb{B}_g(r_g)$ is a connected curve with two endpoints;
2. $r_g \leq N_g/4$;
3. $\mathbb{B}_g(r_g)$ contains no medial axis points on the outer side of Γ;
4. $\alpha_g^L(r) > 0$;
5. $\alpha_g^U(r) \leq \alpha_g^L(r) + 15°$;
6. if p, p', p'' are three points in the given order on a leg, then $\angle(p, p', p'') \geq 175°$;

Note that since Γ is piecewise smooth, for every corner point g, there is such a ball $\mathbb{B}_g(r_g)$. Let α_g^U and α_g^L equal $\alpha_g^U(r_g)$ and $\alpha_g^L(r_g)$, respectively.

Conditions 1–3 on r_g isolate $\mathbb{B}_g(r_g)$ from other corner points or other parts of the curve. Conditions 4–6 on r_g require that the two legs of $\Gamma \cap \mathbb{B}_g(r_g)$ be almost straight.

Let $r'_g = (1 - \sin(\alpha_g^U/2))(r_g/5)$. We will prove that within $\mathbb{B}_g(r'_g)$, the pole directions always point to the outer side of the curve. Note that r'_g depends on the upper bound of the angles at g.

For each $g \in G(\Gamma)$, let δ_g be

$$\min\left(\frac{r_g \sin(\alpha_g^L)}{2}, \frac{r'_g}{4}\right)$$

We will require that our sample set P satisfies sampling condition $R_2(\delta_g)$ within $\mathbb{B}_g(r_g)$ whenever $\alpha_g^U \leq 150°$.

We need a more restrictive sampling condition for the points adjacent to each corner to enable the identification of a corner. Let α be strictly less than the minimum angle at any corner point. Let $\hat{\delta}_g = \min(\delta_g, r_g \sin(\alpha))$. We will require that our sample set P satisfies sampling condition $R_2(\hat{\delta}_g)$ in $\mathbb{B}_g(\hat{\delta}_g)$ whenever $\alpha_g^U \leq 150°$. This implies that g has two sample points within distance $\hat{\delta}_g$ of g and that the edges incident on each of these sample points have length at most $\hat{\delta}_g$. Note that $R_2(\hat{\delta}_g)$ depends both upon the radius of the larger neighborhood around g and upon the global parameter α.

As α_g^U approaches $180°$, r'_g and δ_g approach 0. Thus, when $\alpha_g^U > 150°$, we drop our sampling conditions based on r'_g and δ_g. Instead, we simply require that sample set P satisfies condition $R_2(r_g/2)$ in $\mathbb{B}_g(r_g/2)$ whenever $\alpha_g^U > 150°$.

In the smooth portions of the curve, we will use sampling condition $R_1(0.5)$. We need to overlap this sampling condition and condition $R_2(\delta_g)$ in the neighborhood of corners. Let $r''_g = r'_g - 2\delta_g$ whenever $\alpha_g^U \leq 150°$ and let $r''_g = r_g/2$ whenever $\alpha_g^U > 150°$. We apply sampling condition $R_1(0.5)$ outside of $\cup_{g\in G(\Gamma)}\text{int}(\mathbb{B}_g(r''))$. ($\text{int}(\mathbb{B}_g(r''))$ is the interior of disk $\mathbb{B}_g(r'')$.)

To summarize, our sample set P will satisfies the following sampling conditions:

- For each $g \in G(\Gamma)$ where $\alpha_g^U \leq 150°$, sample set P satisfies sampling condition $R_2(\delta_g)$ in $\mathbb{B}_g(r_g)$;
- For each $g \in G(\Gamma)$ where $\alpha_g^U \leq 150°$, sample set P satisfies sampling condition $R_2(\hat{\delta}_g)$ in $\mathbb{B}_g(\hat{\delta}_g)$;
- For each $g \in G(\Gamma)$ where $\alpha_g^U > 150°$, sample set P satisfies sampling condition $R_2(r_g/2)$ in $\mathbb{B}_g(r_g/2)$;
- Sample set P satisfies sampling condition $R_1(0.5)$ outside of $\cup_{g\in G(\Gamma)}\text{int}(\mathbb{B}_g(r''_g))$.

6. Proof of Correctness

Our proof of correctness is divided into six parts. In the first part we discuss the smooth part of the curve which satisfies sampling condition R_1. The lemmas are

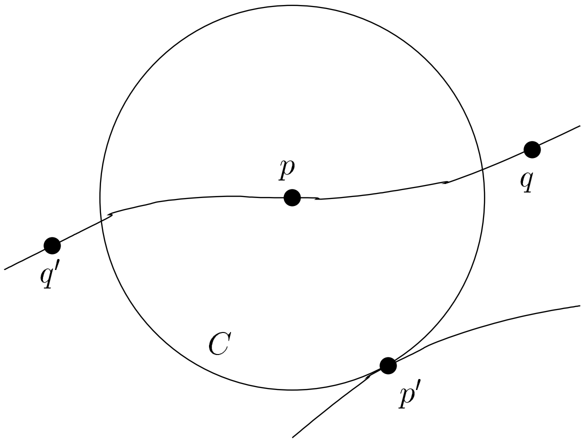


Figure 13: Nearest neighbor of point satisfying sampling condition R_1.

borrowed from Dey and Kumar [9], modified to fit our slightly different definition of sampling.

In the second part, we prove various properties of pole directions in the neighborhood of a sharp corner. In particular, we show that the pole directions of sample points point to the outer side of the curve. The pole direction of a sample point is bounded by the outer normals to the reconstruction edges incident on the sample point. (See Figure 19.) It follows that there is a large angle between the pole directions of two sample points which are not on the same leg.

In the third part, we show that the sampling conditions guarantee that reconstruction edges adjacent to a corner pass the ratio test for ratio $\frac{1}{2\sin(\alpha)}$ on their outer side. We also show that "short" cross edges which are not reconstruction edges do not pass the same ratio test on their outer side.

The fourth part contains two lemmas about the angle between reconstruction and non-reconstruction edges. The fifth part discusses large corner angles where $\alpha_g^U > 150°$. In the final part, we apply these results to show the correctness of our algorithm.

6.1. Sampling Condition R_1

The following lemmas are from Dey and Kumar [9] although the sampling condition is slightly different. Because we need to use and calculate the precise constants under our sampling conditions, we include the proofs.

Lemma 1 *Let P be a sample set of curve Γ which satisfies sampling condition $R_1(1)$ at point $p \in P$. If $p' \in P$ is a nearest neighbor in P of $p \in P$, then edge (p, p') is a correct reconstruction edge of Γ.*

Proof. Assume (p, p') was not a correct reconstruction edge of Γ. Since p satisfies sampling condition $R_1(1)$, the two sample points, q and q', adjacent to p on Γ are at distance at most $f(p)$ from p. Since p' is a nearest neighbor of p, it must be at distance at most $f(p)$ from p.

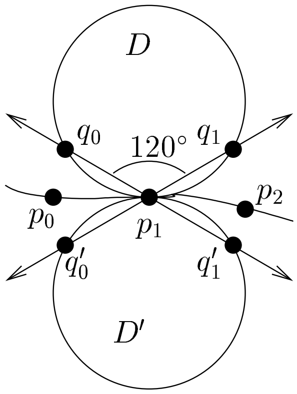


Figure 14: Angle between sample points within distance $f(p_1)$ of p_1.

Draw a circle C of radius $|(p, p')|$ around p. (See Figure 13.) This circle intersects Γ at p' and between p and q and between p and q'. Since the circle intersects Γ at least three times, there is some medial axis point in the interior of this circle. Thus the feature size $f(p)$ at p is less than the radius of the circle which is also $f(p)$, a contradiction. We conclude that (p, p') must be a correct reconstruction edge of Γ. □

Lemma 2 *If p_0, p_1, p_2 are sample points of curve Γ, and p_0 and p_2 lie within distance $f(p_1) > 0$ of p_1, then either $\angle(p_0, p_1, p_2) \geq 120°$ or $\angle(p_0, p_1, p_2) \leq 60°$.*

Proof. Since $f(p_1) > 0$, curve Γ must be smooth at p_1. Let D and D' be the two tangent disks of radius $f(p_1)$ on either side of p_1. Since the feature size of p_1 is $f(p_1)$, disks D and D' do not contain any points of Γ in their interior. (If D or D' contained a point of Γ in their interior, then some smaller tangent disk would touch Γ at two or more points on its boundary and so the feature size would be less than $f(p_1)$.) Let q_0, q_1 be the two points on the boundary of D at distance $f(p_1)$ from p_1. Similarly, let q'_0, q'_1 be the two points on the boundary of D' at distance $f(p_1)$ from p_1. (See Figure 14.) Since p_0 and p_2 lie in a circle of radius $f(p_1)$, they lie in the wedges defined by $\angle(q_0, p_1, q'_0)$ and $\angle(q_1, p_1, q'_1)$. If p_0 and p_2 lie in two different wedges, then $\angle(p_0, p_1, p_2) \geq 120°$. If p_0 and p_2 lie in the same wedges, then $\angle(p_0, p_1, p_2) \leq 60°$. □

Lemma 3 *If P is a sample set of curve Γ which satisfies sampling condition $R_1(0.5)$ at point $p_1 \in P$, then the angle between the two correct reconstruction edges incident on p_1 is greater than 150°.*

Proof. Since P satisfies sampling condition $R_1(0.5)$ at p_1, curve Γ must be smooth at p_1. As in the previous lemma, let D and D' be the two tangent disks of radius $f(p_1)$ on either side of p_1. Let q_0, q_1 and q'_0, q'_1 be the points on the boundaries of D and D', respectively, this time at distance $0.5f(p_1)$ from p_1.

Let p_0 and p_2 be the two sample points adjacent to p_1. Points p_0 and p_2 lie in the wedges defined by $\angle(q_0, p_1, q'_0)$ and $\angle(q_1, p_1, q'_1)$. Let c and c' be the centers of disks D and D', respectively. Since $|(q_0, p_1)| = 0.5f(p_1)$, it follows that $\cos(\angle(q_0, p_1, c)) =$

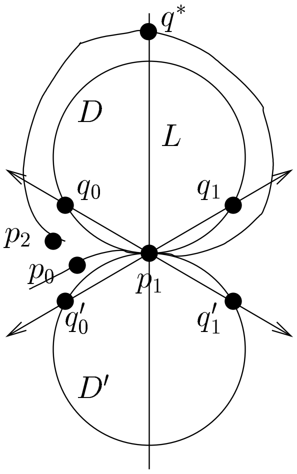


Figure 15: Reconstruction angle.

0.25 and so $\angle(q_0, p_1, c) > 75°$. Similarly, angles $\angle(q_0', p_1, c')$ and $\angle(q_1, p_1, c)$ and $\angle(q_1', p_1, c')$ are all greater than $75°$. Thus $\angle(q_0, p_1, q_1)$ and $\angle(q_0', p_1, q_1')$ are both greater than $150°$.

Let L be the line through p_1 and the centers of D and D'. If p_0 and p_2 lay in the same wedge, then they would lie on the same side of line L and line L would either intersect the interval from p_0 to p_1 or from p_1 to p_2. Let q^* be this intersection point. Since the feature size of p_1 is $f(p_1)$, disks D and D' do not contain any points of Γ in their interior and so q^* cannot lie in D or D'. Thus q^* would have distance at least $2f(p_1)$ from p_1 and so P would not satisfy sampling condition $R_1(0.5)$ at p_1. Since p_0 and p_2 lie in different wedges, the angle between them is greater than $150°$. □

Lemma 4 *If P is a sample set of curve Γ which satisfies sampling condition $R_1(1)$ at point $p_1 \in P$ and q is a sample point which is not adjacent to p_1, then either $|(p_1, q)| > f(p_1)$ or there exists a point p_2 adjacent to p_1 such that $\angle(q, p_1, p_2) \leq 90°$ and $|(p_1, p_2)| < |(p_1, q)|$.*

Proof. Let D be the disk with diameter $|(p_1, q)|$ through p_1 and q. If $D \cap \Gamma$ is not homeomorphic to a line segment, then C contains a medial axis point in its interior and so $|(p_1, q)| > f(p)$. If $D \cap \Gamma$ is homeomorphic to a line segment, then there is some sample point p_2 lying between p_1 and q in $D \cap \Gamma$. Since this point lies in D and (p_1, q) is a diameter of D, it follows that $\angle(q, p_1, p_2) \leq 90°$. □

6.2. Pole Directions

We start our analysis of the pole directions in the neighborhood of a corner point $g \in G(\gamma)$ by showing that poles in a small neighborhood of g must be at least distance $r_g/3$ from their corresponding sample points.

Lemma 5 *Let p be a point in a sample set P of Γ. If p is in $\mathbb{B}_g(r_g/3)$ for some*

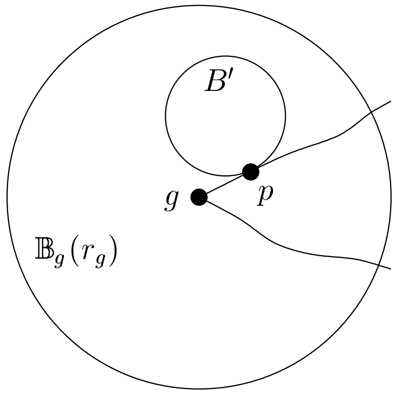


Figure 16: Ball B' tangent to Γ at p.

corner $g \in G(\Gamma)$, then the pole of p is at least $r_g/3$ from Γ.

Proof. Let B' be a ball of radius $r_g/3$ which is tangent to Γ at p and lies on the outer side of Γ. (See Figure 16.) Note that even if p is the corner point g, there is still such a ball B' (actually many such B'.) Since p lies within distance $r_g/3$ of g, B' is contained in $\mathbb{B}(r_g)$. Since the interior of the outer side of Γ does not intersect the medial axis, $int(B')$ does not intersect any other point of the curve and so the center of B' lies in the Voronoi cell of p. Thus the center of B' is a point in the Voronoi cell of p which is at least distance $r_g/3$ from p. $\square$

We will use the following simple trigonometric lemma which can be found in any standard trigonometric text.

Lemma 6 *The radius of the circle through three non-collinear points, p_0, p_1 and p_2, is $\frac{|(p_0,p_2)|}{2\sin(\angle(p_0,p_1,p_2))}$.*

We now show that the pole directions point to the outer side of the curve. For each corner point g, let $r'_g = (1 - \sin(\alpha_g^U/2))(r_g/5)$ and let $\delta_g = \min(r_g \sin(\alpha_g^L)/2, r'_g/4)$. Note that $\delta_g \leq (1 - \sin(\alpha_g^U/2))(r_g/20)$.

Lemma 7 *Let (p, q_1) and (p, q_2) be correct reconstruction edges for a sample set P of curve Γ. If p is in $\mathbb{B}_g(r'_g)$ for some corner $g \in G(\Gamma)$ and P satisfies sampling condition $R_2(\delta_g)$ in $\mathbb{B}_g(r_g)$, then the pole direction of p points to the outer side of curve (q_1, p, q_2).*

Proof. The rays from p to q_1 and p to q_2 partition the plane into two wedges, one on the inner side and one on the outer side of (q_1, p, q_2). (The inner wedge does not necessarily have angle less than 180°.) We wish to show that the the pole of p lies in the wedge on the outer side.

By Lemma 5, the pole of p is at least distance $r_g/3$ from p. Let p' be some point in the intersection of the Voronoi cell of p and the inner wedge defined by (q_1, p, q_2). We will prove that p' is less than $r_g/3$ from p and so cannot be a pole of p.

Let u_1 and u_2 be the furthest points from g on each of the legs of the curve in $\mathbb{B}(r_g)$. By definition of α_g^U, there is a wedge W of angle α_g^U from g which contains all the points in $\Gamma \cap \mathbb{B}_g(r_g)$.

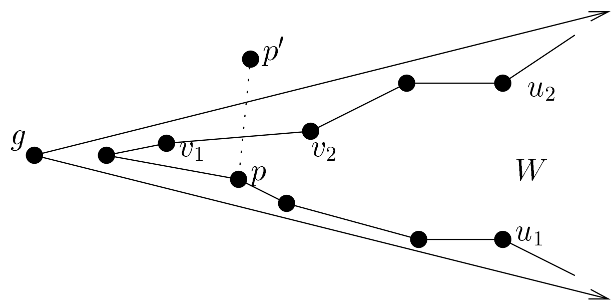


Figure 17: Case I: Point p' lies outside of wedge W.

Case I: Point p' lies outside of wedge W.

By the sampling condition u_1 and u_2 are at least distance $r_g - \delta_g > r_g/3$ from g. Thus the intersection of the perpendicular bisectors of p and u_1 and p and u_2 lies in wedge W. Since p' lies outside of W, line segment (p, p') does not intersect line segment (u_1, u_2). Therefore, line segment (p, p') intersects a correct reconstruction edge (v_1, v_2) where v_1 and v_2 lie between u_1 and u_2 on $\Gamma \cap \mathbb{B}_g(r_g)$. (See Figure 17.)

If $\angle(v_1, p, v_2)$ was less than 90 degrees, then the Voronoi cell of p would not intersect (v_1, v_2) and so line segment (p, p') could not intersect (v_2, v_2). Thus $\angle(v_1, p, v_2)$ must be greater than 90 degrees.

Without loss of generality, assume that v_1 lies between p and v_2 on $\Gamma \cap \mathbb{B}_g(r_g)$. By definition, $\angle(p, v_1, v_2) \geq \alpha_g^L$ and so $\angle(v_1, p, v_2) \leq 180° - \alpha_g^L$. Note that $|(v_1, v_2)| \leq \delta_g \leq r_g \sin(\alpha_g^L)/2$. By Lemma 6, the distance from p to the center of the circle through p, v_1 and v_2 is

$$\begin{aligned} \frac{|(v_1, v_2)|}{2\sin(\angle(v_1, p, v_2))} &\leq \frac{r_g \sin(\alpha_g^L)/2}{2\sin(180° - \alpha_g^L)} \\ &\leq \frac{r_g \sin(\alpha_g^L)/2}{2\sin(\alpha_g^L)} \\ &\leq \frac{r_g}{4} < \frac{r_g}{3}. \end{aligned}$$

Thus the direction from p to p' cannot be the pole direction of p.

Case II: Point p' lies in wedge W.

Let B' be the ball of radius $|(p, p')|$ centered at p'. Ball B' may not be totally contained in W. However, we claim that the ball B'' of radius $|(p, p')| - \delta_g$ centered at p is contained in W. (See Figure 18.)

We first show that $\angle(g, u_1, p') \leq 90°$ and $\angle(g, u_2, p') \leq 90°$. Let u_1' and u_2' be the two points on line segments (g, u_1) and (g, u_2), respectively, which are exactly $r_g - \delta_g$ from g. Let $\tilde{B}$ be the ball tangent to (g, u_1) and (g, u_2) at u_1' and u_2'. The radius of ball $\tilde{B}$ is $(r_g - \delta_g)\tan(\angle(u_1, g, u_2)/2)$ and the distance from g to the center of $\tilde{B}$ is $(r_g - \delta_g)/\cos(\angle(u_1, g, u_2)/2)$. Thus the distance from g to any point on $\tilde{B}$

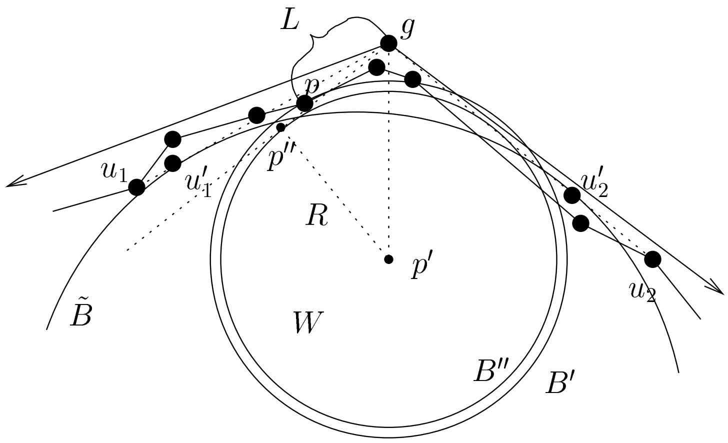


Figure 18: Case II: Point p' lies inside wedge W.

is at least

$$\frac{r_g - \delta_g}{\cos(\frac{\angle(u_1,g,u_2)}{2})} - (r_g - \delta_g)\tan(\frac{\angle(u_1,g,u_2)}{2})) = \frac{(r_g - \delta_g)(1 - \sin(\frac{\angle(u_1,g,u_2)}{2}))}{\cos(\frac{\angle(u_1,g,u_2)}{2})}.$$

Since $\angle(u_1, g, u_2) \leq \alpha_g^U$, this distance is at least

$$\frac{(r_g - \delta_g)(1 - \sin(\alpha_g^U/2))}{\cos(\alpha_g^U/2))}.$$

Let B'' be the ball of radius $|(p,p')| - \delta_g$ around p'. Since u_1' and u_2' are distance δ_g from u_1 and u_2, respectively, and B' does not contain u_1 or u_2, ball B'' does not contain u_1' or u_2'. If $\angle(g, u_1, p')$ or $\angle(g, u_2, p')$ was greater than $90°$, then either $\angle(g, u_1', p')$ or $\angle(g, u_2', p')$ (or both) would be greater than $90°$. Thus ball $\tilde{B}$ would be closer to g than B'' and so the distance from g to B'' is more than

$$\frac{(r_g - \delta_g)(1 - \sin(\alpha_g^U/2))}{\cos(\alpha_g^U/2))}.$$

Since B' differs from B'' by δ_g, the distance from B' to g is more than

$$\begin{aligned}\frac{(r_g - \delta_g)(1 - \sin(\alpha_g^U/2))}{\cos(\alpha_g^U/2))} - \delta_g &\geq \frac{r_g(1 - \sin(\alpha_g^U/2))}{\cos(\alpha_g^U/2))} - 2\delta_g \\ &\geq 5r_g' - 2\delta_g \\ &> r_g'.\end{aligned}$$

(Note that $(1-\sin(\alpha_g^U/2))/\cos(\alpha_g^U/2)$ is less than or equal to 1 for any $\alpha_g^U \leq 180°$.) However, p lies in $\mathbb{B}_g(r_g')$ and p lies on B'. Thus, both $\angle(g, u_1, p')$ and $\angle(g, u_1, p')$ are less than or equal to $90°$.

We now show the ball B'' of radius $|(p,p')| - \delta_g$ centered at p' is contained in W. If not, then there is some point $q \in B''$ which is not contained in W. Since

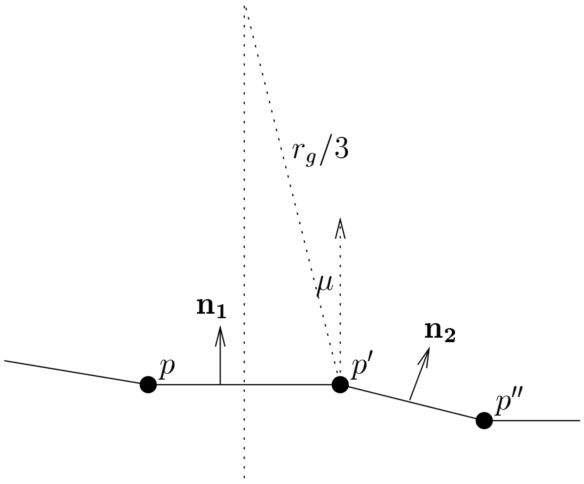


Figure 19: Angle μ between pole direction and edge normals.

$\angle(g, u_1, p') \leq 90°$ and $\angle(g, u_2, p') \leq 90°$, the curve Γ must cross (q, p'). However, ball B' does not contain any sample points, so no sample point lies within distance δ_g of B''. In particular, no sample point lies within distance δ_g of (q, p'). Thus the edge of Γ crossing (q, p') must have length greater than δ_g, violating the sampling condition.

Consider a tangent line to B'' through g intersecting B'' at point p''. Let R be the radius of B'' and let L be the distance from p to g. Since B'' is contained in W, angle $\angle(p', g, p'') \leq \alpha_g^U/2$. Since B'' is within distance δ_g of p, the distance from p' to g is at most $\delta_g + L + R$. Thus, $\sin(\alpha_g^U/2) \geq \frac{R}{R+\delta_g+L}$. Solving for R gives $R \leq \frac{L+\delta_g}{1-\sin(\alpha_g^U/2)}$. Since $L \leq r'_g = r_g(1-\sin(\alpha_g^U/2))/5$ and $\delta_g \leq r_g(1-\sin(\alpha_g^U/2))/20$, it follows that $R \leq r_g/4 < r_g/3$. Thus the direction from p to p' cannot be the pole direction of p.

Since the distance from p to any point p' in the intersection of the Voronoi cell of p and the inner wedge defined by (q_1, p, q_2) is less than $r_g/3$ and the distance from p to its pole is at least $r_g/3$, the pole direction of p must point to the outer wedge defined by (q_1, p, q_2).

□

Not only does the pole direction point toward the outside of the corner, it also approximates the surface normal. To prove this, we show that the pole direction is bounded by the outer normals to its two incident reconstruction edges. Actually, we show that the pole direction is bounded by normals to any "short" edges connecting sample points.

For any two vectors, $\mathbf{u_1}$ and $\mathbf{u_2}$, let $[\mathbf{u_1}, \mathbf{u_2}] +/- \psi°$ denote the set of vectors which either lie in the angle spanned by $\mathbf{u_1}$ and $\mathbf{u_2}$ or make an angle of at most ψ degrees with either $\mathbf{u_1}$ or $\mathbf{u_2}$.

Lemma 8 *Let p, p', p'' be three sample points in sample set P of curve Γ where*

$p' \in \mathbb{B}_g(r_g/3)$ for some corner $g \in G(\Gamma)$. Let $\mathbf{n_1}$ and $\mathbf{n_2}$ be the normals to line segments (p, p') and (p', p''), respectively, on the same side of curve (p, p', p'') as $d(p')$, the pole direction of p'.

1. *If $|(p, p')| \leq \delta_g$ and $|(p', p'')| \leq \delta_g$, then vector $d(p')$ is in the angle spanned by $[\mathbf{n_1}, \mathbf{n_2}] +/- 10°$.*
2. *If $|(p, p')| \leq 2\delta_g$ and $|(p', p'')| \leq 2\delta_g$, then vector $d(p')$ is in the angle spanned by $[\mathbf{n_1}, \mathbf{n_2}] +/- 15°$.*

Proof. Without loss of generality, assume that the outside of curve (p, p', p'') is to the left of directed curve (p, p', p''). Let μ be the maximum angle that $d(p')$ can be turned to the left of $\mathbf{n_1}$. (See Figure 19.)

By Lemma 5, the pole of p' is at least distance $r_g/3$ from p'. If $|(p, p')| \leq \delta_g$, then the perpendicular bisector of (p, p') is at most $\delta_g/2$ from p'. Thus $\sin(\mu) \leq \frac{\delta_g/2}{r_g/3} \leq \frac{1}{12}$ and $\mu < 10°$. If $|(p, p')| \leq 2\delta_g$, then the perpendicular bisector of (p, p') is at most δ_g from p' and $\sin(\mu) \leq \frac{\delta_g}{r_g/3} \leq \frac{2}{12}$ and $\mu < 15°$. The same arguments hold for $\mathbf{n_2}$. Thus, if $|(p, p')| \leq \delta_g$ and $|(p', p'')| \leq \delta_g$, then $d(p')$ lies in the range $[\mathbf{n_1}, \mathbf{n_2}] +/- 10°$. If $|(p, p')| \leq 2\delta_g$ and $|(p', p'')| \leq 2\delta_g$, then $d(p')$ lies in the range $[\mathbf{n_1}, \mathbf{n_2}] +/- 15°$. $\square$

As a direct corollary, the pole directions are bounded by the outer normals of their incident edges.

Lemma 9 *Let P be a sample set of curve Γ which satisfies the sampling condition $R_2(\delta_g)$ on $\Gamma \cap \mathbb{B}_g(r_g)$ for some corner point $g \in G(\Gamma)$. Let $p, p', p'' \in P$ be three adjacent sample points of Γ and let $\mathbf{n_1}$ and $\mathbf{n_2}$ be the outer normals to line segments (p, p') and (p', p''), respectively. If $p' \in \mathbb{B}_g(r'_g)$, then vector $d(p)$, the pole direction of p, is in the angle spanned by $[\mathbf{n_1}, \mathbf{n_2}] +/- 10°$.*

Proof. By Lemma 7, if $p' \in \mathbb{B}_g(r'_g)$, then the pole direction of p' points toward the outside of curve (p, p', p''). Since P satisfies sampling condition $R_2(\delta_g)$, both $|(p, p')|$ and $|(p'', p')|$ are at most δ_g. Applying Lemma 8, vector $d(p')$ lies in the range $[\mathbf{n_1}, \mathbf{n_2}] +/- 10°$. $\square$

Lemma 10 *Let P be a sample set of curve Γ which satisfies the sampling condition $R_2(\delta_g)$ on $\Gamma \cap \mathbb{B}_g(r_g)$ for some corner point $g \in G(\Gamma)$. If $p, p' \in P$ are two (not necessarily adjacent) sample points on the same leg of $\Gamma \cap \mathbb{B}_g(r'_g)$ and neither p nor p' are adjacent to corner g, then $\angle(d(p), d(p')) \leq 30°$.*

Proof. Let e and e' be the correct reconstruction edges incident on p and p', respectively, whose other endpoints do not lie on the portion of $\Gamma \cap \mathbb{B}_g(r_g)$ between p and p'. (See Figure 20.) By the sampling condition, the lengths of e and e' are less than δ_g and so all their endpoints lie in $\mathbb{B}_g(r_g)$. Since p and p' are not adjacent to the corner g, all the endpoints of e and e' lie on the same leg of the curve. By condition 6 on $\Gamma \cap \mathbb{B}_g(r_g)$, the angles between e and (p, p') and between e' and (p, p') are at least 175°. Therefore, the angle between the outer normals of e and e' are at most 10°. Applying Lemma 9, directions $d(p)$ and $d(p')$ lie in the range given by those angles $+/- 10°$. Thus, the angle between $d(p)$ and $d(p')$ is at most 30°. $\square$

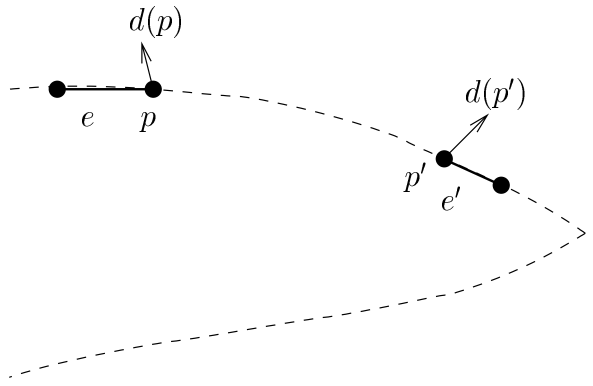


Figure 20: Pole directions of sample points on the same leg of the curve.

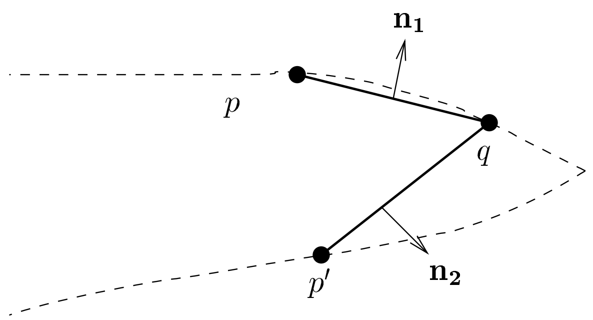


Figure 21: Nearest neighbor of point satisfying sampling condition R_2.

Lemma 11 *Let P be a sample set of curve Γ which satisfies the sampling condition $R_2(\delta_g)$ on $\Gamma \cap \mathbb{B}_g(r_g)$ for some corner point $g \in G(\Gamma)$. If $p, p', p'' \in P$ are three adjacent sample points in $\mathbb{B}_g(r'_g)$ and $\angle(p, p', p'') \leq 30°$, then $\angle(d(p), d(p'')) \geq 130°$.*

Proof. Let $\mathbf{n_1}$ and $\mathbf{n_2}$ be the outer normals to (p, p') and (p', p''), respectively. Since $\angle(p, p', p'') \leq 30°$, the angle between $\mathbf{n_1}$ and $\mathbf{n_2}$ is at least $150°$. Applying Lemma 9, bounds the angle between $d(p)$ and $d(p'')$ by at least $130°$. $\square$

In the neighborhood of a corner, nearest neighbors may not form correct reconstruction edges. However, nearest neighbors whose pole directions are approximately the same do form correct reconstruction edges.

Lemma 12 *Let P be a sample set of curve Γ which satisfies sampling condition $R_2(\delta_g)$ in $\mathbb{B}_g(r_g)$. If $p' \in P$ is a nearest neighbor of point $p \in P$ and $p \in \mathbb{B}_g(r'_g - \delta_g)$ and $\angle(d(p), d(p')) \leq 65°$, then edge (p, p') is a correct reconstruction edge of Γ.*

Proof. Assume (p, p') was not a correct reconstruction edge of Γ. Since $\mathbb{B}_g(r'_g - \delta_g) \subset \mathbb{B}_g(r_g)$, point p satisfies sampling condition $R_2(\delta_g)$. Thus the two sample points adjacent to p are at distance at most δ_g from p. Since p' is a nearest neighbor of p, it also must be at distance at most δ_g from p. Since p lies in $\mathbb{B}_g(r'_g - \delta_g)$,

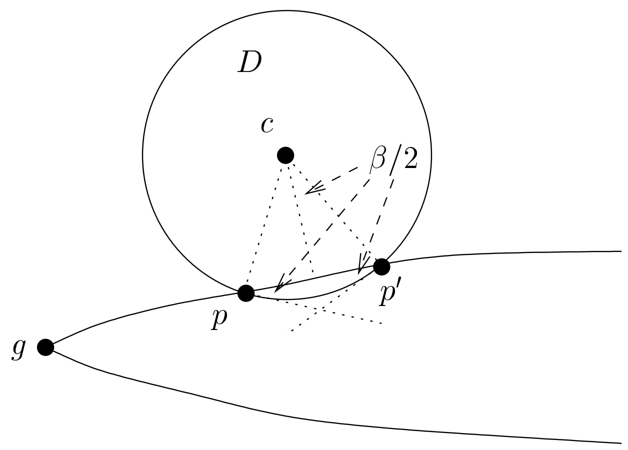


Figure 22: Reconstruction edges in the neighborhood of a corner.

point p' must be be in $\mathbb{B}_g(r'_g)$.

Let q be the sample point adjacent to p which lies between p and p' on $\Gamma \cap \mathbb{B}_g(r_g)$. If p' was on the same leg of $\Gamma \cap \mathbb{B}_g(r_g)$ as p, then condition 6 on $\mathbb{B}_g(r_g)$ guarantees that $\angle(p, q, p') \geq 175°$. In that case, q would be closer to p than p'. Thus, p and p' must lie on different legs of $\Gamma \cap \mathbb{B}_g(r_g)$.

By Lemma 7, the poles at p and p' point to the outside of the curve $\Gamma \cap \mathbb{B}_g(r_g)$. Since $|(p, q)| \geq |(p, p')|$, it follows that $\angle(p, q, p') \leq 90°$. Let $\mathbf{n_1}$ be the outer normal to (p, q) and let $\mathbf{n_2}$ be the normal to (q, p') on the same side of (p, q, p') as $\mathbf{n_1}$. (See Figure 21.) Since $\angle(p, q, p') < 90°$, it follows that $\angle(\mathbf{n_1}, \mathbf{n_2}) > 90°$.

Without loss of generality, assume that the outer side of $\Gamma \cap \mathbb{B}_g(r_g)$ lies to the left of edge (p, q). By Lemma 9, the pole direction of p can be at most $10°$ to the right of $\mathbf{n_1}$. The distance from p to p' is at most δ_g and from p to q is at most δ_g, so the distance from p' to q is at most $2\delta_g$. By Lemma 8, the pole direction of q can be at most $15°$ to the left of $\mathbf{n_2}$. Thus the minimum angle between $d(p)$ and $d(p')$ is greater than $65°$, violating the angle condition on $d(p)$ and $d(p')$. □

Edges connecting nearest neighbors are automatically Delaunay edges of P. We need to show that other edges between adjacent sample points are also Delaunay edges of P.

Lemma 13 *If P is a sample set of curve Γ which satisfies the sampling condition $R_2(\delta_g)$ at sample point $p \in \mathbb{B}(r_g/3)$ and p' is a sample point adjacent to p, then edge (p, p') is in the Delaunay triangulation of P.*

Proof. Let D be the disk of radius $r_g/3$ whose boundary passes through p and p' and whose center lies on the outer side of edge (p, p'). (See Figure 22.) We claim that D contains no sample points other than p and p'.

Since p is at most distance $r_g/3$ from g, disk D is contained in $\mathbb{B}(r_g)$. Thus the portion of D on the outer side of (p, p') contains no medial axis points and thus no sample points.

Let c be the center of disk D and let β equal $\angle(p, c, p')$. Since $|(p, p')| \leq \delta_g \leq$

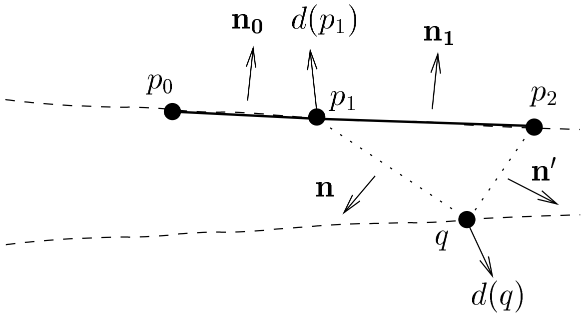


Figure 23: $\angle(d(p_1), d(q)) > 65°$.

$r_g \sin(\alpha_g^L)/2$, it follows that

$$\sin(\beta/2) \leq \frac{\delta_g/2}{r_g/3} \leq \frac{3\sin(\alpha_g^L)}{4}$$

and thus $\beta/2 < \alpha_g^L$.

On the other hand, the tangents to disk D at point p and p' make angles of $\beta/2$ with (p, p'). Thus for any sample point q contained in disk D on the inner side of (p, p'), we have that $\angle(q, p, p') \leq \beta/2 < \alpha_g^L$ and $\angle(q, p', p) \leq \beta/2 < \alpha_g^L$. One of these inequalities violates the definition of α_g^L. Since disk D contains no sample points other than p and p', edge (p, p') is in the Delaunay triangulation of P. $\square$

Finally, we show that if (p_1, q) is a "short" cross edge, then the angle between the pole directions $d(p_1)$ and $d(q)$ is larger than 65°. Procedure Verify_Edge tests the angle between pole directions to avoid "short" cross edges.

Lemma 14 *Let P be a sample set of curve Γ which satisfies sampling condition $R_2(\delta_g)$ in $\mathbb{B}_g(r_g)$, let $p_1 \in P$ be a sample point in $\mathbb{B}_g(r'_g - \delta_g)$ and let $p_0, p_2 \in P$ be the two sample points adjacent to p_1. If $q \in P$ is a sample point not adjacent to p_1, and $\angle(p_0, p_1, q) \geq 120°$ and $|(p_1, q)| \leq |(p_1, p_2)|$, then $\angle(d(p_1), d(q)) > 65°$.*

Proof.

Since P satisfies sampling condition $R_2(\delta_g)$ at p_1, it follows that $|(p_1, q)| \leq |(p_1, p_2)| \leq \delta_g$. Thus point q must also lie in $\mathbb{B}_g(r'_g)$. If q were on the same leg as point p_1, then by condition 6 on $\mathbb{B}_g(r_g)$ either $\angle(p_1, p_2, q) \geq 175°$ or $\angle(p_1, p_2, q) \leq 5°$. The first case is ruled out by $|(p_1, q)| \leq |(p_1, p_2)|$, while the second is elminated by $\angle(p_0, p_1, q) \geq 120°$. Thus q is on a different leg from p_1 or (p_1, q) is a cross edge.

Without loss of generality, assume that the outside of curve (p_0, p_1, p_2) is to the left of directed curve (p_0, p_1, p_2). Let $\mathbf{n_0}$ and $\mathbf{n_1}$ be the outer normals to (p_0, p_1) and (p_1, p_2). By Lemma 9, vector $d(p_1)$ lies in the range $[\mathbf{n_0}, \mathbf{n_1}] +/- 10°$. Let $\mathbf{n}$ be the normal to (p_1, q) on the right side of directed curve (p_0, p_1, q). Let $\mathbf{n'}$ be the normal to (p_2, q) on the left side of directed curve (p_1, p_2, q). (See Figure 23.)

Since $\angle(p_0, p_1, q) \geq 120°$, it follows that $\angle(\mathbf{n_0}, \mathbf{n}) \geq 120°$. Since $|(p_1, q)| \leq |(p_1, p_2)|$, it follows that $\angle(p_1, p_2, q) < 90°$, and thus $\angle(\mathbf{n_1}, \mathbf{n'}) > 90°$. Since $|(p_1, p_2)| \leq \delta_g$ and $|(p_1, q)| \leq \delta_g$, it follows that $|(q, p_2)| \leq 2\delta_g$. By Lemma 8, vector $d(q)$ lies in the range $[\mathbf{n}, \mathbf{n'}] +/- 15°$. Thus, $\angle(d(p_1), d(q)) > 65°$. $\square$

6.3. Ratio Test

The ratio test in procedure Shortest_Potential() is a key element in the rest of the algorithm. We show by the sampling conditions that an edge whose endpoint is adjacent to a corner passes this ratio test and that a cross edge which is shorter than adjacent reconstruction edges fails this test.

As defined before, α is some angle which is strictly less than the minimum angle at any corner point and $\hat{\delta}_g = \min(\delta_g, r_g \sin(\alpha))$.

Lemma 15 *If P is a sample set of curve Γ which satisfies the sampling condition $R_2(\hat{\delta}_g)$ at sample point $p \in P$ adjacent to corner point $g \in G(\Gamma)$ and p' is a sample point adjacent to p, then edge (p, p') passes the ratio test on its outer side for ratio* $1/(2\sin(\alpha))$.

Proof. Let D be the disk of radius $(r_g - \hat{\delta}_g)/2$ whose boundary passes through p and p' and whose center lies on the outer side of edge (p, p'). (See Figure 22.) We claim that D contains no sample points other than p and p'. The proof is similar to the proof of Lemma 13.

Since p is adjacent to g and satisfies sampling condition $R_2(\hat{\delta}_g)$, the distance from the p to g is at most $\hat{\delta}_g$. Since p lies on disk D, disk D is contained in $\mathbb{B}_g(r_g)$. Thus, the part of D on the outer side of (p, p') does not contain any sample points other than p and p'.

By the sampling condition, the length of (p, p') is at most $\hat{\delta}_g \leq \delta_g \leq r_g \sin(\alpha_g^L)/2 < r_g/2$. Let c be the center of disk D and let β equal $\angle(p, c, p')$. Since $|(p, p')| \leq \hat{\delta}_g$, it follows that

$$\sin(\beta/2) \leq \frac{\hat{\delta}_g/2}{(r_g - \hat{\delta}_g)/2} = \frac{\hat{\delta}_g}{r_g - \hat{\delta}_g} < \frac{\hat{\delta}_g}{r_g/2} \leq \frac{2r_g \sin(\alpha_g^L)/2}{r_g} = \sin(\alpha_g^L)$$

and thus $\beta/2 < \alpha_g^L$.

On the other hand, the tangents to disk D at point p and p' make angles of $\beta/2$ with (p, p'). Thus for any sample point q contained in disk D on the inner side of (p, p'), we have that $\angle(q, p, p') \leq \beta/2 < \alpha_g^L$ and $\angle(q, p', p) \leq \beta/2 < \alpha_g^L$. One of these inequalities violates the definition of α_g^L. Thus disk D contains no sample points other than p and p'.

The ratio of the radius of D to the length of edge e is at least

$$\frac{(r_g - \hat{\delta}_g)/2}{\hat{\delta}_g} \geq \frac{r_g}{r_g \sin(\alpha)} - \frac{1}{2} \geq \frac{1}{2\sin(\alpha)}.$$

Thus (p, p') passes the ratio test on its outer side for ratio $1/(2\sin(\alpha))$. □

As explained in Section 5, the outer side is defined for any line segment connecting two points in $\Gamma \cap \mathbb{B}_g(r_g)$. We show that "short" cross edges fail the ratio test on their outer side.

Lemma 16 *Let p and p' be sample points from sample set P on two different legs of $\Gamma \cap \mathbb{B}_g(r_g)$. If $q \in P$ is a sample point lying between p and p' on $\Gamma \cap \mathbb{B}_g(r_g)$ and $|(p, p')| \leq |(p, q)|$, then line segment (p, p') fails the ratio test on its outer side for ratio* $1/(2\sin(\alpha))$.

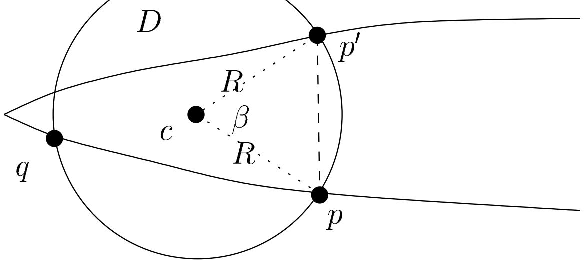


Figure 24: Cross edge in the neighborhood of a corner.

Proof. Since $|(p,p')| \leq |(p,q)|$, it follows that $\angle(p,q,p') \leq 90°$. Thus $\alpha_g^L \leq 90°$ and so $\alpha_g^U \leq 120°$ by condition 5 on $\mathbb{B}_g(r_g)$. Applying condition 6 on $\mathbb{B}_g(r_g)$, shows that q must be on the outer side of (p,p').

Let D be the disk whose boundary passes through p, p' and q. (See Figure 24.) Let c be the center and R the radius of disk D. By definition of α and α_g^L, we have $\angle(p,q,p') \geq \alpha_g^L > \alpha$. Let β equal $\angle(p,c,p')$. The length of (p,p') is $2R\sin(\beta/2)$. Since β equals $2\angle(p,q,p') > 2\alpha$, the length of (p,p') is greater than $2R\sin(\alpha)$. Thus the ratio of the radius of D to the length of (p,p') is less than $1/(2\sin(\alpha))$.

Any disk larger than D containing p and p' whose center is on the outside of (p,p') must also contain q in its interior. Thus, line segment (p,p') fails the ratio test on its outer side for ratio $1/(2\sin(\alpha))$. □

Our algorithm calls a procedure Verify_Edge to verify that an edge is a reconstruction edge. We must show that almost all the reconstruction edges in the neighborhood of a corner pass this test and that no non-reconstruction edge does so.

If (p_1,p_2) is a correct reconstruction edge and neither p_1 nor p_2 is adjacent to a corner, then by Lemma 10 the angle between $d(p_1)$ and $d(p_2)$ is at most $30° < 60°$. By condition 6 on $\Gamma \cap B_g(r_g)$, if p_1 is not adjacent to a corner, then the angle between the two edges incident on p_1 is at most $175° > 120°$. Thus, if neither p_1 nor p_2 is adjacent to a corner, then Verify_Edge will correctly verify (p_1,p_2) as a reconstruction edge.

If p_2 is adjacent to a corner, then $d(p_1)$ and $d(p_2)$ can vary greatly. We show that under appropriate conditions Verify_Edge will correctly verify (p_1,p_2) even if p_2 is adjacent to a corner.

Lemma 17 *Let P be sample set of curve Γ which satisfies the sampling condition $R_2(\delta_g)$ in $\mathbb{B}_g(r_g)$. If $p_0,p_1,p_2,p_3 \in P$ are four adjacent sample points in $\mathbb{B}_g(r'_g)$, and $\mathbf{n}$ is the outer normal to (p_1,p_2) and $\angle(p_0,p_1,p_2) \geq 175°$ and $\angle(p_1,p_2,p_3) \geq 85°$, then $\angle(d(p_1),d(p_2)) \leq 120°$ and $\angle(\mathbf{n},d(p_2)) \leq 120°$.*

Proof. Let $\mathbf{n}'$ be the outer normal to (p_2,p_3). (See Figure 25.) Since $\angle(p_1,p_2,p_3) \geq 85°$, it follows that $\angle(\mathbf{n},\mathbf{n}') \leq 95°$. By Lemma 9, $d(p_2)$ lies in the angle spanned by $[\mathbf{n},\mathbf{n}'] +/- 10°$. Thus $d(p_2)$ lies within $105°$ of $\mathbf{n}$.

Let $\mathbf{n}'$ be the outer normal to (p_0,p_1). Since $\angle(p_0,p_1,p_2) \geq 175°$, it follows that

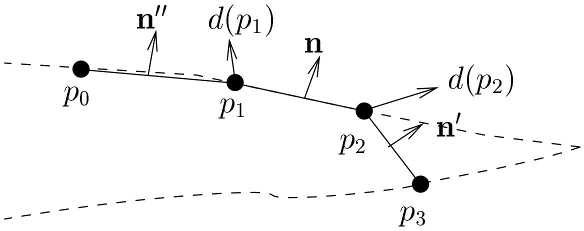


Figure 25: If $\angle(p_0, p_1, p_2) \geq 175°$ and $\angle(p_1, p_2, p_3) \geq 85°$, then $\angle(d(p_1), d(p_2)) \leq 120°$ and $\angle(\mathbf{n}, d(p_2)) \leq 120°$.

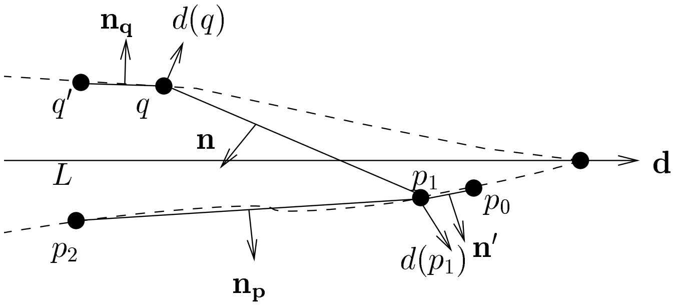


Figure 26: Non-reconstruction cross edge.

$\angle(\mathbf{n}, \mathbf{n}'') \leq 5°$. thus $\angle(\mathbf{n}'', \mathbf{n}') \leq 100°$. Since by Lemma 9, $d(p_1)$ lies in the angle spanned by $[\mathbf{n}, \mathbf{n}''] + / - 10°$ and $d(p_2)$ lies in the angle spanned by $[\mathbf{n}, \mathbf{n}'] + / - 10°$, the angle between $d(p_1)$ and $d(p_2)$ is at most $120°$. □

If p_0, p_1, p_2, p_3 are four adjacent sample points and p_2 is adjacent to a corner while p_1 is not, then $\angle(p_0, p_1, p_2) \geq 175°$ by condition 6 on $\Gamma \cap B_g(r_g)$. If $\angle(p_1, p_2, p_3) \geq 85°$, then $\angle(d(p_1), d(p_2)) \leq 120°$ and $\angle(\mathbf{n}, d(p_2)) \leq 120°$ by Lemma 17. If P satisfies sampling condition $R_2(\hat{\delta})$ at p_2, then (p_2, p_3) passes the ratio test on its outer side for ratio $1/(2\sin(\alpha))$ by Lemma 15. Thus, an edge with one vertex p_2 adjacent to a corner will pass the verification test as long as the angle at p_2 is not too small.

We need also to show that non-reconstruction edges will not pass the verification test.

Lemma 18 *Let P be a sample set of curve Γ which satisfies sampling condition $R_2(\delta_g)$ in $\mathbb{B}_g(r_g)$, let $p_1 \in P$ be a sample point in $\mathbb{B}_g(r'_g - \delta_g)$ and let $p_0, p_2 \in P$ be the two sample points adjacent to p_1. If $q \in P$ is a sample point closer to p_1 than p_2 and $\angle(p_0, p_1, q) \geq 120°$ and $\mathbf{n}$ is the normal to edge (p_1, q) on the same side of curve (p_0, p_1, p_2) as $d(p_1)$ then either:*

- *$\angle(d(p_1), d(q)) > 120°$, or*
- *$\angle(d(q), \mathbf{n}) > 120°$, or*
- *edge (p_1, q) fails the ratio test on side $\mathbf{n}$ for ratio $1/2(\sin(\alpha))$.*

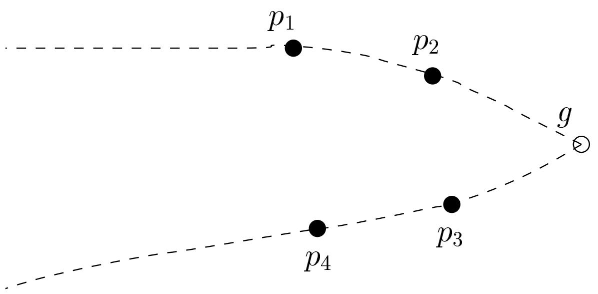


Figure 27: Either $\angle(p_1, p_2, p_3) > 85°$ or $\angle(p_2, p_3, p_4) > 85°$.

Proof. Since P satisfies sampling condition $R_2(\delta_g)$ at p_1, points p_0, p_2 and q must also lie in $\mathbb{B}_g(r'_g)$. If q was on the same leg of $\Gamma \cap \mathbb{B}_g(r_g)$ as p_1, then $\angle(p_1, p_2, q) \geq 175°$ by condition 6 on $\mathbb{B}_g(r_g)$ and thus q would not be closer to p_1 than p_2. Therefore, q must be on a different leg of $\Gamma \cap \mathbb{B}_g(r_g)$ from p_1.

If $\mathbf{n}$ points to the outer side of edge (p_1, q), then edge (p_1, q) fails the ratio test on side $\mathbf{n}$ for ratio $1/2(\sin(\alpha))$.

Assume $\mathbf{n}$ points to the inner side of edge (p_1, q). (See Figure 26.) Let q' be the sample point adjacent to q which does not lie between q and p_1 on $\Gamma \cap \mathbb{B}(r_g)$. Let $\mathbf{n_q}$ be the outer normal to (q, q') and let $\mathbf{n_p}$ be the outer normal to (p_1, p_2). Let L be a line through the corner point g which separates p_1 from q. Let $\mathbf{d}$ be the direction of L pointing away from the cone defined by $\angle(q, g, p_1)$. Condition 6 on $\mathbb{B}_g(r_g)$ states that $\angle(g, q, q') \geq 175°$ and $\angle(g, p_1, p_2) \geq 175°$. Thus $\angle(\mathbf{d}, \mathbf{n_p}) \leq 95°$ and $\angle(\mathbf{d}, \mathbf{n_q}) \leq 95°$.

Either $\mathbf{d}$ is contained in the cone defined by $d(p_1)$ and $d(q)$ or it is not. If it is not contained in that cone, then $\angle(d(p_1), d(q)) \geq 360° - (95° + 95°) = 170°$, proving the assertion. Assume $\mathbf{d}$ is contained in that cone. Let $\mathbf{n}'$ be the outer normal to (p_0, p_1). Since $\angle(p_0, p_1, q) \geq 120°$, it follows that $\angle(\mathbf{n}, \mathbf{n}') \leq 60°$. Applying Lemma 9, the angle between $\mathbf{n}$ and $d(p_1)$ is at most $70°$. If $\angle(d(q), \mathbf{n})$ was less than or equal to $120°$, then the angle between $d(p_1)$ and $d(q)$ would be at least $360° - (70° + 120°) = 170° > 120°$, again proving the assertion. □

6.4. Reconstruction Angles

We need two theorems about the angles between adjacent reconstruction edges in the neighborhood of corners. First, we claim that in the neighborhood of a corner at most one such angle is less than $85°$. Secondly, a "short" cross edge (p, p') which is not a reconstruction edge makes an angle at most $125°$ with one of the reconstruction edges incident on p or p'.

Lemma 19 *If P is a sample set of Γ, then there is at most one set of three adjacent points p, p', p'' lying in $\mathbb{B}_g(r_g)$ such that $\angle(p, p', p'') \leq 85°$.*

Proof. By condition 6 on $\mathbb{B}_g(r_g)$, if $\angle(p, p', p'') \leq 85°$, then p' must be adjacent to a corner. Let p_1, p_2, p_3, p_4 be four adjacent sample points where p_2 and p_3 are adjacent to the corner g. (See Figure 27.) By condition 6 on $\mathbb{B}_g(r_g)$,

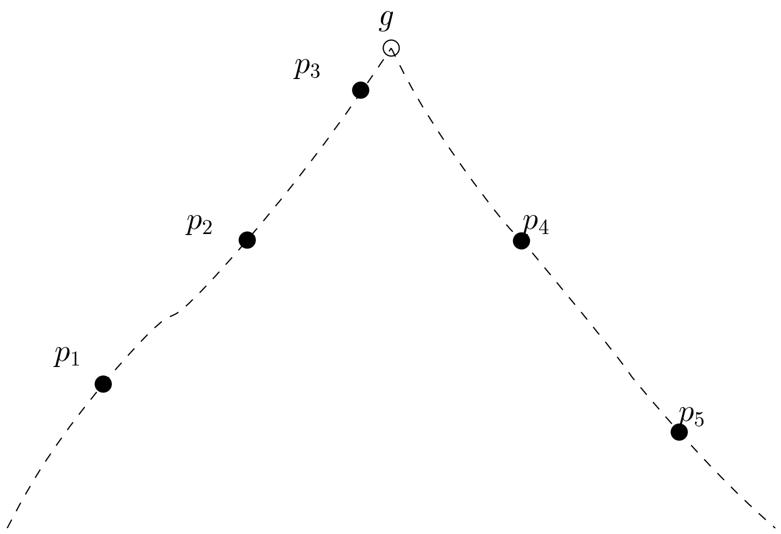


Figure 28: Either $|(p_2, p_4) > |(p_2, p_3)|$ or $\angle(p_1, p_2, p_4) \leq 125°$ or $\angle(p_5, p_4, p_2) \leq 125°$.

both $\angle(p_1, p_2, g) \geq 175°$ and $\angle(p_4, p_3, g) \geq 175°$. If $\angle(p_1, p_2, p_3) \leq 85°$, then $\angle(p_3, p_2, g) \geq 90°$. If $\angle(p_2, p_3, p_4) \leq 85°$, then $\angle(p_2, p_3, g) \geq 90°$. However, $\angle(p_3, p_2, g) + \angle(p_2, p_3, g) \leq 180°$ and so both cannot be greater than $90°$. $\square$

Lemma 20 *Let p_1, p_2, p_3, p_4, p_5 be five sample points of Γ lying in the given order on $\Gamma \cap \mathbb{B}_g(r_g)$. If $|(p_2, p_4)| \leq |(p_2, p_3)|$, then either $\angle(p_1, p_2, p_4) \leq 125°$ or $\angle(p_5, p_4, p_2) \leq 125°$.*

Proof. If p_2 and p_4 were on the same leg of Γ then $\angle(p_2, p_3, p_4) \geq 175°$ and $|(p_2, p_4)| > |(p_2, p_3)|$. So assume p_2 and p_4 are on different legs of Γ. (See Figure 28.)

Assume $\angle(p_1, p_2, p_4) > 125°$ and $\angle(p_5, p_4, p_2) > 125°$. Without loss of generality, assume p_3 lies on the same leg as p_2. By condition 6 on $\mathbb{B}_g(r_g)$, $\angle(p_1, p_2, p_3) \geq 175°$ and so $\angle(p_4, p_2, p_3) \leq 185° - \angle(p_1, p_2, p_4) < 60°$. Similarly, $\angle(p_5, p_4, g) \geq 175°$ and so $\angle(p_2, p_4, p_3) \leq \angle(p_2, p_4, g) \leq 185° - \angle(p_5, p_4, p_2) < 60°$. Thus $\angle(p_2, p_3, p_4) > 60° > \angle(p_2, p_4, p_3)$ and so $|(p_2, p_3)| < |(p_2, p_4)|$. $\square$

6.5. Large Corner Angles

If $\alpha_g^U > 150°$, then we apply sampling condition $R_2(r_g/2)$ in $\mathbb{B}_g(r_g)$. We need two lemmas about reconstruction edges in such neighborhoods. Note that $\alpha_g^U > 150°$ implies that $\alpha_g^L > 135°$ by condition 5 on $\mathbb{B}_g(r_g)$.

Lemma 21 *If sample points p and p' lie in $\mathbb{B}_g(r_g)$ where $\alpha_g^L \geq 90°$ and p' is not adjacent to p, then there is a sample point q adjacent to p such that $|(p, q)| < |(p, p')|$ and $\angle(q, p, p') \leq 90°$.*

Proof. Since p' is not adjacent to p, then there is a sample point $q \in \mathbb{B}_g(r_g)$ lying between p and p' on $\Gamma \cap \mathbb{B}_g(r_g)$. Applying $\alpha_g^L \geq 90°$ gives $\angle(p, q, p') \geq 90°$ and thus $|(p, q)| \leq |(p, p')|$. Since $\angle(p, q, p') \geq 90°$, it must be that $\angle(q, p, p') \leq 90°$. $\square$

As an immediate corollary, if the nearest neighbor to p is inside $\mathbb{B}_g(r_g)$, then the nearest neighbor is adjacent to p.

Corollary 1 *If sample points p and p' lie in $\mathbb{B}_g(r_g)$ where $\alpha_g^L \geq 90°$ and p' is the nearest neighbor of p, then edge (p, p') is a correct reconstruction edge of Γ.*

6.6. Algorithm Correctness

We are finally ready for the main theorem showing that, under appropriate sampling conditions, algorithm Sharp_Reconstruction correctly reconstructs sharp curves. We need to partition the correct reconstruction edges into certain subsets. For each corner point $g \in G(\Gamma)$ where $\alpha_g^U \leq 150°$, let $\mathbb{E}_g$ be the set of correct reconstruction edges both of whose endpoints lie in $\mathbb{B}_g(r'_g - \delta_g)$. Let $\mathbb{E}'_g \subseteq \mathbb{E}_g$ be the set of reconstruction edges in $\mathbb{B}_g(r'_g - \delta_g)$ whose endpoints lie on the same leg of $\Gamma \cap \mathbb{B}_g(r_g)$ and whose adjacent reconstruction edges form angles of at least 85°. Note that if the two endpoints of an edge in $\mathbb{E}_g$ are not adjacent to the corner, then both angles are at least 175° by condition 6 on $\mathbb{B}_g(r_g)$ and so the edge is in $\mathbb{E}'_g$. Note also that since by Lemma 19 there can be at most one sample point in $\mathbb{B}_g(r_g)$ whose incident edges form an angle less than 85° and this sample point must be adjacent to the corner g, there are at most two edges of $\mathbb{E}_g$ missing from $\mathbb{E}'_g$.

Theorem 1 *If P be is a sample set of curve Γ such that:*

- *For each $g \in G(\Gamma)$ where $\alpha_g^U \leq 150°$, sample set P satisfies sampling condition $R_2(\delta_g)$ in $\mathbb{B}_g(r_g)$;*
- *For each $g \in G(\Gamma)$ where $\alpha_g^U \leq 150°$, sample set P satisfies sampling condition $R_2(\hat{\delta}_g)$ in $\mathbb{B}_g(\hat{\delta}_g)$;*
- *For each $g \in G(\Gamma)$ where $\alpha_g^U > 150°$, sample set P satisfies sampling condition $R_2(r_g/2)$ in $\mathbb{B}_g(r_g/2)$;*
- *Sample set P satisfies sampling condition $R_1(0.5)$ outside of $\cup_{g \in G(\Gamma)} \mathrm{int}(\mathbb{B}_g(r''_g))$;*

then algorithm Sharp_Reconstruction correctly reconstructs Γ from P.

Proof. Algorithm Sharp_Reconstruction starts by calling procedure Connect_Nearest_Neighbors. Every sample point p either satisfies sampling condition $R_1(0.5)$ or lies in $\mathbb{B}_g(r'_g - \delta_g)$ for some $\alpha_g^U \leq 150°$ and satisfies sampling condition $R_2(\delta_g)$ or lies in $\mathbb{B}_g(r_g/2)$ for $\alpha_g^U > 150°$. If point p lies in $\mathbb{B}_g(r_g/2)$ for $\alpha_g^U > 150°$, then it satisfies sampling condition $R_2(r_g/2)$ and its nearest neighbor lies in $\mathbb{B}_g(r_g)$. By Lemmas 1, 12 and Corollary 1, every pair of nearest neighbors whose pole directions lie within 60° of one another form a correct reconstruction edge. Thus, Connect_Nearest_Neighbors does not erroneously mark any non-reconstruction edges as reconstruction edges.

We claim that certain reconstruction edges are found by Connect_Nearest_Neighbors. Let p be a point in $\mathbb{B}_g(r'_g - \delta_g)$ but outside $\mathbb{B}_g(r'_g - 2\delta_g)$. Let p' be the nearest neighbor of p. Since p is outside $\mathbb{B}_g(r'_g - 2\delta_g)$, sampling condition $R_1(0.5)$ is satisfied at p. By Lemma 1, edge (p, p') is a reconstruction edge. Since p is inside $\mathbb{B}_g(r_g)$, sampling conditions $R_2(\delta_g)$ is satisfied at p and thus $|(p,p')| \leq \delta_g$. Thus p' in inside $\mathbb{B}_g(r'_g)$ and outside of $\mathbb{B}_g(r'_g - 3\delta_g)$. Since $r'_g > 4\delta_g$, neither p nor p' are adjacent to the corner g. Thus, by Lemma 10, $\angle(d(p), d(p')) \leq 30°$ and so Connect_Nearest_Neighbors finds the reconstruction edge (p,p').

Sharp_Reconstruction next calls procedure Extend_Pole_Pole. We claim that Extend_Pole_Pole does not erroneously mark any non-reconstruction edges as reconstruction edges. We also claim that Extend_Pole_Pole finds all reconstruction edges in $\mathbb{E}'_g$.

Assume that Extend_Pole_Pole does mark a non-reconstruction edge as a reconstruction edge. Let (p_1, q) be the first such non-reconstruction edge. Let p_0 and p_2 be the sample points adjacent to p_1, with (p_0, p_1) already marked as a reconstruction edge.

Assume that p_1 lies outside of $\cup_{g \in G(\Gamma)} \text{int}(\mathbb{B}_g(r''_g))$. Point set P satisfies sampling condition $R_1(0.5)$ at p_1. We claim that (p_1,p_2) passes all the tests in Shortest_Potential. Since (p_1, q) is the first incorrectly marked non-reconstruction edge, at most one edge incident on p_2 is marked as a reconstruction edge. By Lemma 3, $\angle(p_0, p_1, p_2) > 150°$ and so (p_1, p_2) satisfies all the other tests in Shortest_Potential.

By Lemma 4, $|(p_1,p_2)| < |(p_1,q)|$. Since (p_1,p_2) satisfies all the tests in Shortest_Potential and $|(p_1,p_2)| < |(p_1,q)|$, edge (p_1,q) could not be returned by Shortest_Potential and so could not have been marked as a reconstruction edge by Extend_Pole_Pole.

Next assume $p_1 \in \mathbb{B}_g(r_g/2)$ for some $g \in G(\Gamma)$ where $\alpha_g^U > 150°$. Point set P satisfies sampling condition $R_2(r_g/2)$ at p_1. We again claim that (p_1, p_2) passes all the tests in Shortest_Potential. Since (p_1, q) is the first incorrectly marked non-reconstruction edge, at most one edge incident on p_2 is marked as a reconstruction edge. Since $\alpha_g^U > 150°$, it follows from condition 5 on $\mathbb{B}_g(r_g)$ that $\alpha_g^L > 135°$ and so $\angle(p_0,p_1,p_2) > 135°$. Thus (p_1,p_2) satisfies all the other tests in Shortest_Potential.

By the sampling condition, $|(p_1,p_2)| \leq r_g/2$. If $|(p_1,q)| > r_g/2$, then $|(p_1,q)| > |(p_1,p_2)|$. If $|(p_1,q)| \leq r_g/2$, then $q \in \mathbb{B}_g(r_g)$. By Corollary 1, $|(p_1,p_2)| < |(p_1,q)|$. Since (p_1,p_2) satisfies all the tests in Shortest_Potential and $|(p_1,p_2)| < |(p_1,q)|$, edge (p_1,q) could not be returned by Shortest_Potential and so could not have been marked as a reconstruction edge by Extend_Pole_Pole.

Now assume $p_1 \in \mathbb{B}_g(r'_g - 2\delta_g)$ for some $g \in G(\Gamma)$ where $\alpha_g^U \leq 150°$. Again we claim that (p_1,p_2) passes all the tests in Shortest_Potential. Since (p_1, q) is the first incorrectly marked non-reconstruction edge, at most one edge incident on p_2 is marked as a reconstruction edge. By Lemma 11, if (p_1,p_2) makes an angle of 30° or less with reconstruction edge (p_0,p_1), then $\angle(d(p_0), d(p_2)) \geq 130° \geq 90°$. Similarly, if (p_1,p_2) makes an angle of 30° or less with a reconstruction edge (p_2,p_3), then $\angle(d(p_1), d(p_3)) \geq 130° \geq 90°$. Finally, if (p_1,p_2) makes an angle of 135° or less with a reconstruction edge, then one of its endpoints is adjacent to a corner g. By Lemma 15, edge (p_1,p_2) passes the ratio test for ratio $1/(2\sin(\alpha))$ on its outer side. Since (p_1,p_2) passes all the tests in Shortest_Potential but is not returned as the shortest potential edge, the length of (p_1,q) must be less than the length of (p_1,p_2).

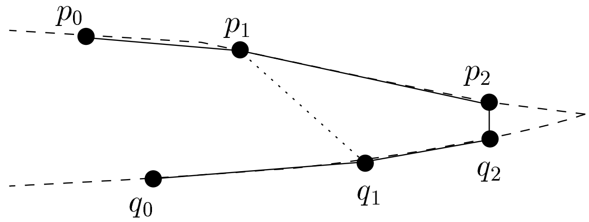


Figure 29: $(q_0, q_1) \prec (p_1, p_2)$ and $\angle(q_0, q_1, p_1) \leq 135°$ so (p_1, q_1) will be eliminated as a potential reconstruction edge.

If q was on the same leg of $\Gamma \cap \mathbb{B}_g(r_g)$ as p_1, then $\angle(p_1, p_2, q) \geq 175°$ by condition 6 on $\mathbb{B}_g(r_g)$ and thus q would not be closer to p_1 than p_2. Therefore, q must be on a different leg of $\Gamma \cap \mathbb{B}_g(r_g)$ from p_1. By Lemmas 14 and 18, edge (p_1, q) will fail to be verified by procedure Verify_Edge. Therefore, Extend_Pole_Pole will not mark (p_1, q) as a reconstruction edge.

To show that Extend_Pole_Pole reconstructs all edges in $\mathbb{E}'_g$, we need to order those edges. For any two edges $e, e' \in \mathbb{E}'_g$ on the same leg of $\mathbb{B}_g(r'_g) \cap \Gamma$, we say that $e \prec e'$ if e' lies between e and g. For any two edges $e, e' \in \mathbb{E}'_g$ on different legs of $\mathbb{B}_g(r'_g) \cap \Gamma$, we say that $e \prec e'$ if there is some cross edge (p, p') such that p separates e from g and e' separates p' from g. This gives a total ordering on all the reconstruction edges in $\mathbb{E}'_g$.

We claim that Extend_Pole_Pole finds all reconstruction edges in $\mathbb{E}'_g$ (which have not yet been found by Connect_Nearest_Neighbor.) Assume not. Let (p_1, p_2) be the first such reconstruction edge not found using the total ordering described above.

Without loss of generality, assume that p_2 lies between p_1 and g. (See Figure 29.) Let (p_0, p_1) be the other reconstruction edge incident on p_1. We claim that (p_0, p_1) has already been reported as a reconstruction edge. If (p_0, p_1) is in $\mathbb{E}'_g$, then $(p_0, p_1) \prec (p_1, p_2)$ and so by assumption must have already been reported. If (p_0, p_1) is not in $\mathbb{E}'_g$, then p_0 must lie outside of $\mathbb{B}_g(r'_g - \delta_g)$. Point set P satisfies sampling condition $R_2(\delta_g)$ at p_1, so $|(p_0, p_1)| \leq \delta_g$. Thus p_1 lies inside $\mathbb{B}_g(r'_g - \delta_g)$ but outside $\mathbb{B}_g(r'_g - 2\delta_g)$ As noted above, Connect_Nearest_Neighbor would report one of the reconstruction edges incident on p_1, either (p_0, p_1) or (p_1, p_2). Since (p_1, p_2) is not found by Connect_Nearest_Neighbor or Extend_Pole_Pole, Connect_Nearest_Neighbor must have found (p_0, p_1).

We claim that Shortest_Potential will return edge (p_1, p_2). If not, there must be some cross edge (p_1, q_1) shorter than (p_1, p_2) which was returned by Shortest_Potential. By Lemma 16, edge (p_1, q_1) fails the ratio test on its outer side. Therefore, it must be that $\angle(p_0, p_1, q_1) > 135°$, so that Shortest_Potential does not apply the ratio test to (p_1, q_1).

Let $q_0 \in P$ be the sample point adjacent to q_1 which does not lie between q_1 and g. Edge (q_0, q_1) precedes (p_1, p_2) in our total ordering. We show that $(q_0, q_1) \in E'$. Since q_0 is not adjacent to the corner, the angle between the edges incident to q_0 is at least $175° \geq 85°$. Let q_2 be the other sample point adjacent to q_1. If q_1 is not adjacent to a corner, then $\angle(q_0, q_1, q_2) \geq 175° \geq 85°$ and so $(q_0, q_1) \in E'$. However, if q_1 is adjacent to a corner, then the argument is a little more complicated.

Since $|(p_1, q_1)| \leq |(p_1, p_2)|$,

$$\angle(p_1, p_2, q_1) \leq \angle(p_1, q_1, p_2) \leq \angle(q_0, q_1, p_2) \leq \angle(q_0, q_1, q_2).$$

If $\angle(q_0, q_1, p_2) < 85°$ and $\angle(p_1, p_2, q_1) < 85°$, then either $\angle(q_0, q_1, g) < 175°$ or $\angle(p_1, p_2, g) < 175°$ violating condition 6 on $\Gamma \cap \mathbb{B}_g(r_g)$. Thus $\angle(q_0, q_1, q_2) \geq 85°$ and so $(q_0, q_1) \in \mathbb{E}'_g$.

Since $(q_0, q_1) \in \mathbb{E}'_g$ and $(q_0, q_1) \prec (p_1, p_2)$, edge (q_0, q_1) must already have been identified as a reconstruction edge. Since $\angle(p_0, p_1, q_1) \geq 135°$, it follows by Lemma 20 that $\angle(q_0, q_1, p_1) \leq 125° < 135°$. Thus (p_1, q_1) fails the third test in Shortest_Potential and so cannot be the edge returned by Shortest_Potential.

We have shown that Shortest_Potential returns the edge (p_1, p_2). If p_2 is not adjacent to a corner, then Verify_Edge correctly identifies (p_1, p_2) as a reconstruction edge by Lemma 10. If p_2 is adjacent to a corner, then sampling condition $R_2(\hat{\delta}_g)$ is satisfied at p_2. Verify_Edge correctly identifies (p_1, p_2) as a reconstruction edge by Lemmas 15 and 17. Thus Extend_Pole_Pole finds all the reconstruction edges in $\mathbb{E}'_g$.

The next call is to procedure Add_Corner_Edges. Add_Corner_Edges replaces the call to Verify_Edge with a 70° angle test. We claim that Add_Corner_Edges does not erroneously add any non-reconstruction edges outside of $\cup_{g \in G(\Gamma)} \mathbb{B}_g(r'_g - 2\delta_g)$, and that it correctly finds all remaining unreported edges in $\mathbb{E}_g - \mathbb{E}'_g$ for each $g \in G(\Gamma)$.

Let $p_1 \in P$ be some sample point outside of $\cup_{g \in G(\Gamma)} \mathbb{B}_g(r'_g - 2\delta_g)$ and let p_0 and p_2 be the sample points adjacent to p_1 with (p_0, p_1) already marked as a reconstruction edge. As argued above, (p_1, p_2) passes all the tests in Shortest_Potential. Assume $|(p_1, q)| \leq |(p_1, p_2)$ for some sample point $q \in P$. Since P satisfies sampling condition $R_1(1)$ at p_1, it follows that $|(p_1, q)| \leq |(p_1, p_2)| \leq f(p)$. By Lemma 2, either $\angle(p_0, p_1, q) \leq 60°$ or $\angle(p_0, p_1, q) \geq 120°$. If $\angle(p_0, p_1, q) \leq 60° < 70°$, edge (p_1, q) would not be added by Add_Corner_Edges. If $\angle(p_0, p_1, q) \geq 120°$, then, by Lemma 4, $|(p_1, p_2)| < |(p_1, q)|$, contradicting the assumption that $|(p_1, q)| \leq |(p_1, p_2)|$. Thus Add_Corner_Edges does not erroneously add any non-reconstruction edges outside of $\cup_{g \in G(\Gamma)} \mathbb{B}_g(r'_g - 2\delta_g)$.

Now let $p_1 \in P$ be some sample point in $\mathbb{B}_g(r_g/2)$ where $\alpha_g^U > 150°$. Let p_0 and p_2 be the sample points adjacent to p_1 with (p_0, p_1) already marked as a reconstruction edge. As argued above, (p_1, p_2) passes all the tests in Shortest_Potential. Assume $|(p_1, q)| \leq |(p_1, p_2)$ for some sample point $q \in P$. Since $|(p_1, p_2)| \leq r_g/2$, length $|(p_1, q)| \leq r_g/2$ and q lies in $\mathbb{B}_g(r_g)$. Since $\alpha_g^L \geq 135°$, either $\angle(p_0, p_1, q) \geq 135°$ or either $\angle(p_0, p_1, q) \leq 45°$. If $\angle(p_0, p_1, q) \leq 45° < 70°$, edge (p_1, q) would not be added by Add_Corner_Edges. If $\angle(p_0, p_1, q) \geq 135°$, then, by Lemma 21, $|(p_1, p_2)| < |(p_1, q)|$, contradicting the assumption that $|(p_1, q)| \leq |(p_1, p_2)|$. Thus Add_Corner_Edges does not erroneously add any non-reconstruction edges where $\alpha_g^U > 150°$.

We claim that Add_Corner_Edges correctly reconstructs all the remaining edges in $\mathbb{B}_g(r'_g - \delta_g)$ where $\alpha_g^U \leq 150°$. There are at most two edges in $\mathbb{E}_g - \mathbb{E}'_g$ and so at most two edges which have not been found. These edges are incident on a vertex which is adjacent to the corner. Since sample set P satisfies sampling condition $\hat{\delta}_g$ within $\mathbb{B}_g(\hat{\delta}_g)$, these edges have length at most $\hat{\delta}_g$. By Lemma 15, they pass the ratio test on their outer side for ratio $1/(2\sin(\alpha))$.

If there is only one edge, (p, p'), which has not been found, then Shortest_Potential will return that edge when called with parameter p and with parameter p'. Either the angle at p or the angle at p' is at least $85° \geq 70°$, and so (p, p') will be found.

Assume there are two edges, (p, p') and (p', p''), which have not been found. The angles at p and at p'' must be at least $85° \geq 70°$. If $|(p, p'')| \leq |(p, p')|$, then by Lemma 16 it will fail the ratio test on its outer side for ratio $1/(2\sin(\alpha))$. By Lemma 20, edge (p, p'') will make an angle at most $125° < 135°$ with one of the two discovered reconstruction edges incident on p or p''. Thus edge (p, p'') will fail the Shortest_Potential test and edges (p, p') and (p', p'') will be found.

The final procedure, Extend_Smooth, is exactly Dey and Kumar's algorithm. It completes the reconstruction in the smooth portions of the curve and in $\mathbb{B}_g(r_g)$ where $\alpha_g^U > 150°$. By Lemma 3, the angle between two reconstruction edges (p_0, p_1) and (p_1, p_2) in the smooth portions of the curve is greater than $150° > 90°$. In $\mathbb{B}_g(r_g)$ where $\alpha_g^U > 150°$, this angle is at least $\alpha_g^L > 135°$. By Lemmas 4 and 21, any other edge which makes an angle greater than $90°$ with (p_0, p_1) will be longer than (p_1, p_2). $\square$

7. Running Time Analysis

Computing the Voronoi diagram of n sample points takes $O(n \log n)$. Construct_Nearest_Neighbors takes time proportional to the number of edges in the Delaunay triangulation which is $O(n)$. Extend_Smooth adds a vertex p to stack S at most 3 times, once initially and once when each reconstruction edge incident on p is identified. Thus Extend_Smooth runs in $O(n)$ time.

Extend_Pole_Pole may add a vertex many times to the stack since it adds all the neighbors of p_1 and p_2 to the stack when it identifies p_1 and p_2 as a reconstruction edge. If p is a neighbor of p_1, then we can charge this addition of p to the stack to edge (p, p_1). Since only two edges incident on p_1 are identified as reconstructed edges, edge (p, p_1) is only charged twice. Point p is also added on the stack when each reconstruction edge incident on p is identified and it is initially placed in the stack. Thus if m is the number of Delaunay edges, then point p is added to the stack at most $2m + 3n \in O(n)$ times. Note that this analysis and the previous analysis of Extend_Smooth is independent of whether Sharp_Reconstruction correctly reconstructs the curves.

At each iteration, Extend_Pole_Pole may call Shortest_Potential. A naive implementation of Shortest_Potential might check all the edges incident on p_1. However, such a check may take $\Omega(n)$ time giving an $\Omega(n^2)$ worst case running time. Instead, one should presort the edges incident on a given vertex by length and keep an index of the current shortest potential edge. As each edge (p, p') is marked as

Connect_Nearest_Neighbors2($\mathcal{T}$, α)
/* Connect nearest neighbors */

for each vertex p_1 of $\mathcal{T}$ **do**
  $p_2 \leftarrow$ Nearest-Neighbor(p_1);
  if (p_1, p_2).long_nonrecon = **false then**
    if both $d(p_1)$ and $d(p_2)$ form an angle at least $45°$ with edge (p_1, p_2) **then**
      if $\angle(d(p_1), d(p_2)) \leq 60°$, **then**
        Mark (p_1, p_2) as a reconstruction edge;
      else if $\angle(d(p_1), d(p_2)) \leq 120°$ and (p_1, p_2) passes the ratio test on both sides for ratio 1, **then**
        Mark (p_1, p_2) as a reconstruction edge;
      else if (p_1, p_2) passes the ratio test on both sides for ratio $1/(2\sin(\alpha))$, **then**
        Mark (p_1, p_2) as a reconstruction edge;
      endif
endfor

Figure 30: Modified connect nearest neighbors algorithm.

a reconstruction edge, check whether (p, p') causes any of the edges incident on p and on p' to no longer be a potential edge. Mark such edges as non-reconstruction edges and update the shortest potential edges of their endpoints, if necessary. A non-reconstruction edge is adjacent to four reconstruction edges so it is processed at most four times. By keeping the incident edges in sorted order, one never need revisit any edges shorter than the current shortest potential edge in updating the shortest potential edge. The presorting takes $O(n \log n)$ while the rest of the processing takes $O(n)$ time. Thus Extend_Pole_Pole can be implemented to run in $O(n \log n)$. Add_Corner_Edges is simply a modification of Extend_Pole_Pole and also runs in $O(n \log n)$ time. Thus Sharp_Reconstruction runs in $O(n \log n)$ time.

8. Implementation

We have presented an algorithm which corrected reconstructs a family of closed curves under appropriate sampling conditions. However, sampling conditions may be violated and the curves may not be closed. In this section we discuss practical modifications to our algorithm which improve its performance under these conditions. We are careful that these modifications retain the reconstruction guarantees of the original algorithm.

Procedure Connect_Nearest_Neighbors connects nearest neighbors whose pole directions are approximately the same. The pole test prevents the algorithm from connecting non-adjacent nearest neighbors in the neighborhood of a corner. (See Figure 1.) However, if an edge passes the ratio test on both sides for ratio $1/(2\sin(\alpha))$, then, by Lemma 16, it cannot connect such non-adjacent nearest neighbors. Thus we modify Connect_Nearest_Neighbors to also connect nearest neighbors which pass the ratio test for ratio $1/(2\sin(\alpha))$ on both sides.

The ratio test for ratio $1/(2\sin(\alpha))$ depends on the minimum corner angle α. For $\alpha = 10°$, this ratio is approximately three which makes the ratio test fairly difficult to pass. The following lemma shows that if the angle between pole directions is at most 120°, then "short" non-reconstruction cross edges will fail the ratio test for ratio 1.

Lemma 22 *Let p and p' be sample points from sample set P on two different legs of $\Gamma \cap \mathbb{B}_g(r_g)$. If $q \in P$ is a sample point lying between p and p' on $\Gamma \cap \mathbb{B}_g(r_g)$ and $|(p,p')| \leq |(p,q)|$ and $\angle(d(p), d(p')) \leq 120°$, then line segment (p,p') fails the ratio test on its outer side for ratio 1.*

As a corollary, reconstruction edges can be detected by a combination of the pole and ratio test.

Lemma 23 *Let P be a sample set of curve Γ which satisfies sampling condition $R_2(\delta_g)$ in $\mathbb{B}_g(r_g)$. If $p' \in P$ is a nearest neighbor of $p \in \mathbb{B}_g(r'_g - \delta_g)$ and $\angle(d(p), d(p')) \leq 120°$ and (p,p') passes the ratio test for ratio 1 on both sides, then edge (p,p') is a correct reconstruction edge of Γ.*

The proof idea is that if p and p' are in the neighborhood of corner g and $\angle(d(p), d(p')) \leq 120°$, then the angle at g must be at least 30°. Using this lower bound on the angle at g, any "short" non-reconstruction cross edge in the neighborhood of g must fail the ratio test on its outer side for ratio 1. We modify Connect_Nearest_Neighbors to also connect nearest neighbors whose pole directions lie within 120° of each other and which pass the ratio test for ratio 1 on both sides.

An isolated point from "noise" can confuse Connect_Nearest_Neighbors since it will try to connect such a point to a true sample point on the surface. We reduce interference from such isolated points, by adding a requirement in Connect_Nearest_Neighbors that the pole directions be almost perpendicular to the reconstruction edge. The final version is in Figure 30.

If p is the nearest neighbor of p' and p'' is the closest point to p' such that $\angle(p, p', p'') \geq 90°$, then (p', p'') is a good reconstruction candidate. Such edges are added by Dey and Kumar's curve reconstruction algorithm [9] for smooth closed curves. We would like to also add such edges after applying the same tests we use for nearest neighbors. However, such tests may fail to detect non-reconstruction edges if point p' is adjacent to a sharp corner or adjacent to the endpoint of a curve. We add an additional requirement that (p', p'') not be "too long" compared to (p, p'). We require that (p', p'') is at most twice the length of (p, p') which seems appropriately conservative. This test is dependent on the sampling distribution but it does not invalidate any guarantees our algorithm makes under appropriate sampling. The new procedure is called Connect_Close. (See Figure 31.)

Undersampling at the corners can cause a true reconstruction edge to fail the ratio test for ratio $1/(2\sin(\alpha))$ in procedure Shortest_Potential. We modify this test in a number of ways. First, we apply a weak ratio test with ratio 0.5. If the angle with the current reconstruction edge is less than 90° or if the potential edge is incident on two reconstruction edges and the angle with both is less than 135°,

Connect_Close($\mathcal{T}$, α)
/* Connect close neighbors */

1. **for** each vertex p_1 of $\mathcal{T}$ **do**
2. $p_0 \leftarrow$ Nearest-Neighbor(p_1);
3. $p_2 \leftarrow$ Closest point to p_1 such that $\angle(p_0, p_1, p_2) \geq 90°$;
4. **if** (p_1, p_2).long_nonrecon = **false then**
5. **if** $|(p_1, p_2)| \leq 2|(p_0, p_1)|$ **then**
6. **if** both $d(p_1)$ and $d(p_2)$ form an angle at least 45° with edge (p_1, p_2) **then**
7. **if** $\angle(d(p_1), d(p_2)) \leq 60°$, **then**
8. Mark (p_1, p_2) as a reconstruction edge;
9. **else if** $\angle(d(p_1), d(p_2)) \leq 120°$ and (p_1, p_2) passes the ratio test on both sides for ratio 1, **then**
10. Mark (p_1, p_2) as a reconstruction edge;
11. **else if** (p_1, p_2) passes the ratio test on both sides for ratio $1/(2\sin(\alpha))$, **then**
12. Mark (p_1, p_2) as a reconstruction edge;
13. **endif**

Figure 31: Connect close neighbors.

Shortest_Potential2($\mathcal{T}$, p, α)
/* return shortest potential edge incident on p */

1. $(q_0, q_1) \leftarrow$ shortest edge incident on p such that for each endpoint q_i of e:
    - At most one edge incident on q_i is marked as a reconstruction edge;
    - If q_i is incident on a reconstruction edge (q_i, q), and $\angle(q_{1-i}, q_i, q) \leq 30°$, then $\angle(d(q_{1-i}), d(q)) \geq 90°$;
      /* Note q_{1-i} is the other endpoint of e */
    - If q_i is incident on a reconstruction edge (q_i, q) and $\angle(q_{1-i}, q_i, q) \leq 135°$ then Vor_Del_Ratio($\mathcal{T}$, q, q_i, q_{1-i}) ≥ 0.5;
    - If q_i is incident on a reconstruction edge (q_i, q) and $\angle(q_{1-i}, q_i, q) \leq 90°$ OR if q_i and q_{1-i} are incident on reconstruction edges (q_i, q) and (q_i, q') respectively and $\angle(q_{1-i}, q_i, q) \leq 135°$ and $\angle(q_i, q_{1-i}, q') \leq 135°$ then
        - Vor_Del_Ratio($\mathcal{T}$, q, q_i, q_{1-i}) ≥ 1;
        - if $\angle(d(q_i), d(q_{1-i})) \leq 120°$ AND either $\angle(d(q_i), \mathbf{n}) \geq 120°$ or $\angle(d(q_{1-i}), \mathbf{n}) \geq 120°$, then Vor_Del_Ratio($\mathcal{T}$, q, q_i, q_{1-i}) $\geq \frac{1}{2\sin(\alpha)}$;
2. **return**((q_0, q_1));

Figure 32: Modified criteria for shortest potential edges.

```
Verify_Edge2(𝒯, p0, p1, p2)
/* verify that (p1,p2) is a reconstruction edge */
/* edge (p0,p1) is an identified reconstruction edge */
if (p1,p2).long_nonrecon = false then
        if ∠(p0,p1,p2) ≥ 120° then
                if ∠(d(p1),d(p2)) ≤ 60° then
                        return (true); /* verified reconstruction edge */
                n ← normal to (p1,p2) on same side of curve (p0,p1,p2) as d(p1);
                if (∠(d(p1),d(p2)) ≤ 120°) and (∠(n,d(p2)) ≤ 120°) then
                        if Vor_Del_Ratio(𝒯, p0, p1, p2) ≥ 1 then
                                return (true); /* verified reconstruction edge */
endif
return ( false ); /* unable to verify reconstruction edge */
```

Figure 33: Modified edge verification.

then we apply a stronger ratio test with ratio 1.0. Finally, if the angle with the current reconstruction edge is less than $120°$ and some angle between the two pole directions and the edge outer normal is greater than $120°$, then we apply the full ratio test with ratio $1/(2\sin(\alpha))$. (See Figure 32.) Lemma 22 can be used to justify this combination pole and ratio test.

Procedure Verify_Edge (Figure 8) uses the ratio test with ratio $1/(2\sin(\alpha))$. However, by applying Lemma 22, we can actually justify reducing that ratio to 1. (See Figure 33.)

Open curves present a challenge for our algorithm. The problem is that Extend_Pole_Pole or Add_Corner_Edge can misinterpret a curve endpoint as the leg of some curve near a corner and add a long edge from that endpoint. There is little we can do about this in Extend_Pole_Pole. However, a corner edge should be significantly shorter than any Delaunay edges which point away from the corner. We can check this condition whenever we attempt to add an edge in Add_Corner_Edges.

The Voronoi-Delaunay ratio test is based on the minimum corner angle α. If this angle α is small, then Extend_Pole_Pole and Add_Corner_Edges may fail to add edges near undersampled corners. To try and retrieve these edges, we rerun Extend_Pole_Pole and Add_Corner_Edges with α replaced by $30°$. Properly sampled corners will already have been reconstructed and will be unaffected by this attempt.

We modify Extend_Smooth in a number of ways to handle undersampling. Instead of searching over all neighbors, we only consider neighbors which are not already incident on two reconstruction edges and which make an angle of at least $120°$ with the current reconstruction edge. We find the closest such neighbor and check that it makes an angle of at least $150°$ with the current reconstruction edge to guarantee that we are reconstructing the smooth portion of the curve.

As a final post processing step, we check our reconstruction for inappropriate "long" edges. These are edges whose distance to the nearest curve endpoint is smaller than half their length. We also reapply the Check_Corner_Edge test to any

```
Check_Corner_Edge($\mathcal{T}$, $p_0$, $p_1$)
/* check that edge $(p_0, p_1)$ is "short" */
for $i = 0$ to 1 do
        if $p_i$ is incident on a reconstruction edge $(p_i, q)$ other than $(p_0, p_1)$ then
                $q_i \leftarrow p_i$;
                $q'_i \leftarrow q$;
        else
                $q_i \leftarrow p_{1-i}$;
                $q'_i \leftarrow p_i$;
        endif
endfor
return (Check_Corner_Edge_Alg($\mathcal{T}$, $p_0$, $p_1$, $q_0$, $q'_0$, $q_1$, $q'_1$));
```

```
Check_Corner_Edge_Alg($\mathcal{T}$, $p_0$, $p_1$, $q_0$, $q'_0$, $q_1$, $q'_1$))
/* check that edge $(p_0, p_1)$ is "short" */
/* $(q_0, q'_0)$ and $(q_1, q'_1)$ are first edges on the curve legs */
/* Directions $(q'_0 - q_0)$ and $(q'_1 - q_1)$ point away from the corner */
for $i = 0$ to 1 do
        for each Delaunay edge $(q, p_i)$ do
                if $\angle(q_0, q'_0, q) \geq 70°$ and $\angle(q_1, q'_1, q) \geq 70°$ then
                        if $|(p_0, p_1)| > 2|(q, p_i)|$ then
                                return (false);
                        endif
return ( true );
```

Figure 34: Check corner edge.

```
Add_Corner_Edges2($\mathcal{T}$, $\alpha$)
/* Add edges adjacent to a corner */
Add all vertices to stack $S$;
while $S \neq \emptyset$ do
        $p_1 \leftarrow S.Pop()$;
        if $p_1$ is incident on exactly one reconstruction edge $(p_0, p_1)$ then
                $(p_1, p_2) \leftarrow$ Shortest_Potential($\mathcal{T}$, $p_1$, $\alpha$);
                if $(p_1, p_2)$.long_nonrecon = false then
                        if $\angle(p_0, p_1, p_2) \geq 70°$ then
                                if Check_Corner_Edge($\mathcal{T}$, $p_1$, $p_2$) then
                                        Mark $(p_1, p_2)$ as a reconstruction edge;
                                        Add $p_2$ and all neighbors of $p_1$ and $p_2$ to stack
                                        $S$;
endwhile
```

Figure 35: Modified add corner edges algorithm.

```
Extend_Smooth2(T)
/* Add remaining edges in smooth portions of the curve */
for each vertex p1 of T do
    p2 ← Nearest-Neighbor(p1);
    if p1 and p2 are not incident on any reconstruction edges, then
        Mark (p1, p2) as a reconstruction edge;
    endif
endfor
Add all vertices to stack S;
while S ≠ ∅ do
    p1 ← S.Pop();
    if p1 is incident on exactly one reconstruction edge (p0, p1) then
        (p1, p2) ← shortest edge incident on p1 such that ∠(p0, p1, p2) ≥
        120° and p2 is incident on fewer than two reconstruction edges;
        if (p1, p2).long_nonrecon = false then
            if ∠(p0, p1, p2) ≥ 150 then
                Mark (p1, p2) as a reconstruction edge;
                Add p2 to stack S;
            endif
endwhile
```

Figure 36: Modified extend smooth portions of the curve.

```
Flag_Long_Nonrecon(T)
/* Flag any long non-reconstruction edges */
for each edge reconstruction edge (p0, p1) do
    for each edge Delaunay edge (q, pj) do
        if q not incident on two reconstruction edges and
        |(q, pj)| < |(p0, p1)|/2 then
            (p0, p1).long_nonrecon ← true;
    endfor
    if (p0, p1) makes a reconstruction angle less than 105° then
        if Check_Corner_Edge(T, (p0, p1)) = false then
            (p0, p1).long_nonrecon ← true;
endfor
```

Figure 37: Identify and flag long non-reconstruction edges.

GathanG(P, n, α, m)
/* P is a set of n sample points */
/* α is a strict lower bound on the minimum corner angle */
/* m is the number of iterations permitted */
1. Construct the Delaunay triangulation, $\mathcal{T}$, of P;
2. **for** each edge e of $\mathcal{T}$ **do**
3. e.long_nonrecon ← **false**;
4. **endfor**
5. **repeat**
6. GathanG_Alg($\mathcal{T}, \alpha$);
7. Flag_Long_Nonrecon($\mathcal{T}$);
8. **for** m iterations **or** until no additional edges are flagged as long non-reconstruction edges;

GathanG_Alg($\mathcal{T}, \alpha$)
/* $\mathcal{T}$ is the Delaunay triangulation of the sample points */
/* α is a strict lower bound on the minimum corner angle */
1. Connect_Nearest_Neighbors2($\mathcal{T}$); /* Connect nearest neighbors */
2. Connect_Close($\mathcal{T}$, α); /* Connect close neighbors */
3. Extend_Pole_Pole($\mathcal{T}$, α); /* Extend using pole-pole and angle tests */
4. Add_Corner_Edges2($\mathcal{T}$, α); /* Add edges adjacent to corners */
5. Extend_Pole_Pole($\mathcal{T}$, 30°); /* Extend using pole-pole and angle tests */
6. Add_Corner_Edges2($\mathcal{T}$, 30°); /* Add edges adjacent to corners */
7. Extend_Smooth2($\mathcal{T}$); /* Extend to smooth portions of the curve */

Figure 38: Curve reconstruction algorithm.

curve next to a corner. If we find a long edge which fails these tests then we flag it as a long non-reconstruction edge and rerun our curve reconstruction from the start. In each procedure, we check whether an edge is so flagged directly before adding it to our set of reconstruction edges.

In most of our examples, this final post processing step is not needed. In the other examples, our algorithm correctly reconstructed the curve in the second iteration.

We claim although do not prove that none of these modifications invalidate the reconstruction guarantee in Theorem 1. The final version of our algorithm we call GathanG for Gathan with guarantees since it is a "guaranteed" version of our heuristic Gathan described in [8]. An implementation can be downloaded from //www.cis.ohio-state.edu/graphics.

9. Results

We present the results of running GathanG on a number of sample data sets

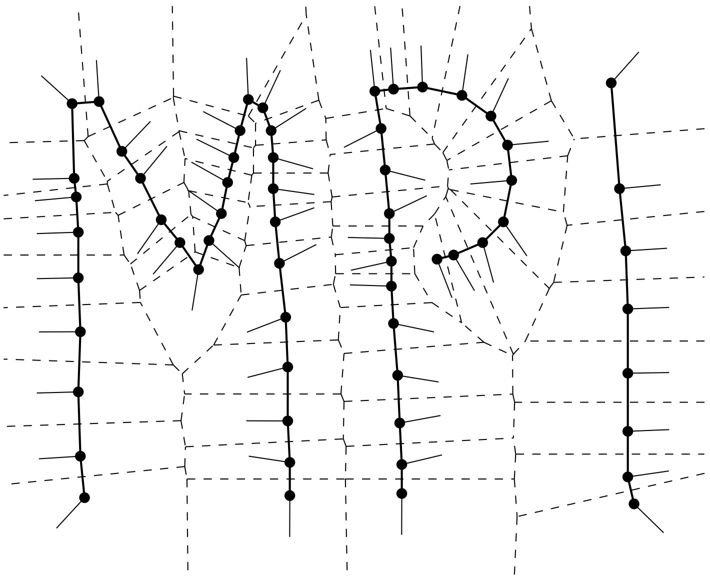


Fig. 39. Voronoi diagram, pole directions and reconstruction of MPI data set [13].

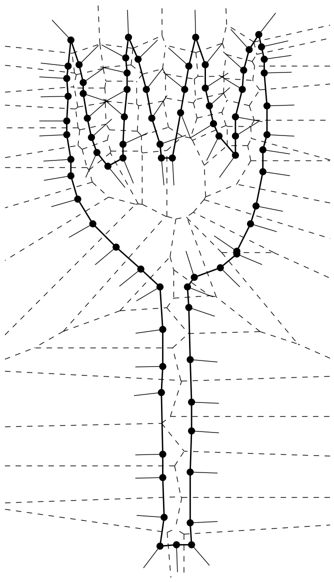


Fig. 40. Voronoi diagram, pole directions and reconstruction of tulip data set[2] (79 sample points.)

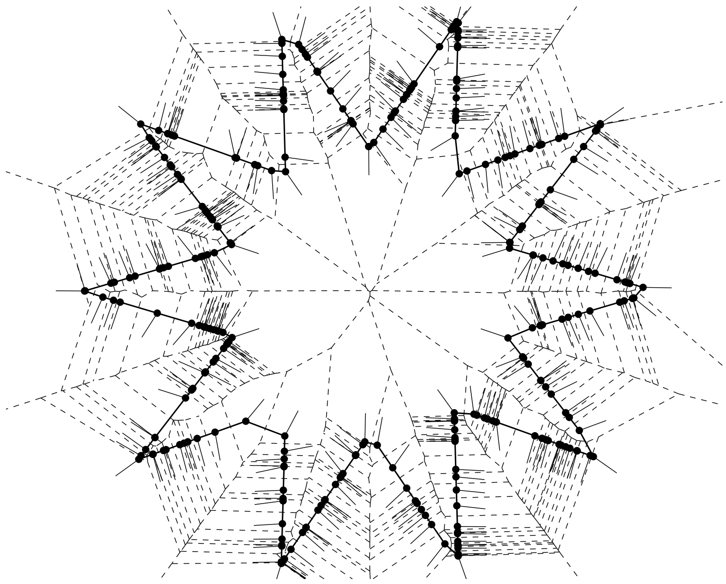


Fig. 41. Voronoi diagram, pole directions and reconstruction of star data set[2], 10 spikes, 300 sample points.

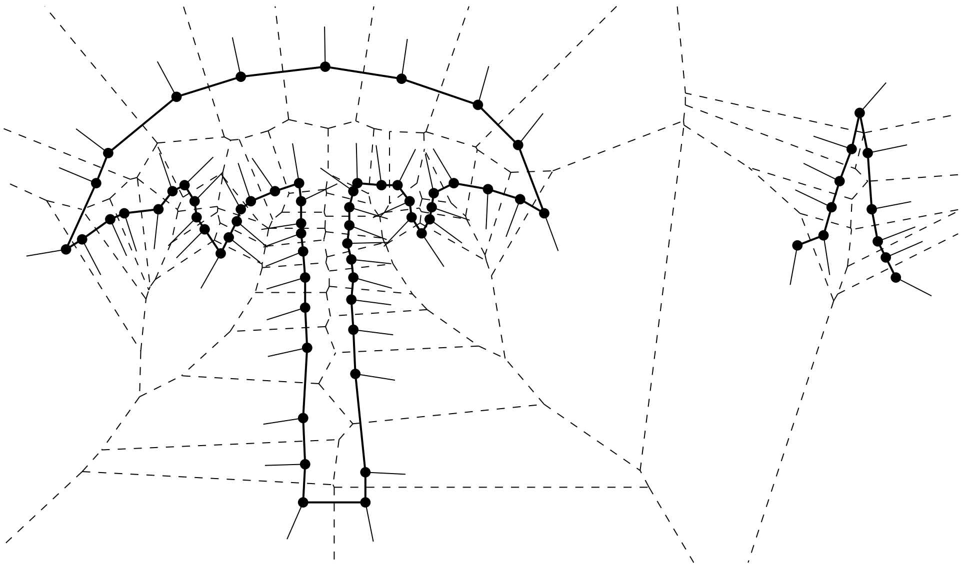


Fig. 42. Voronoi diagram, pole directions and reconstruction of mushroom data set[8].

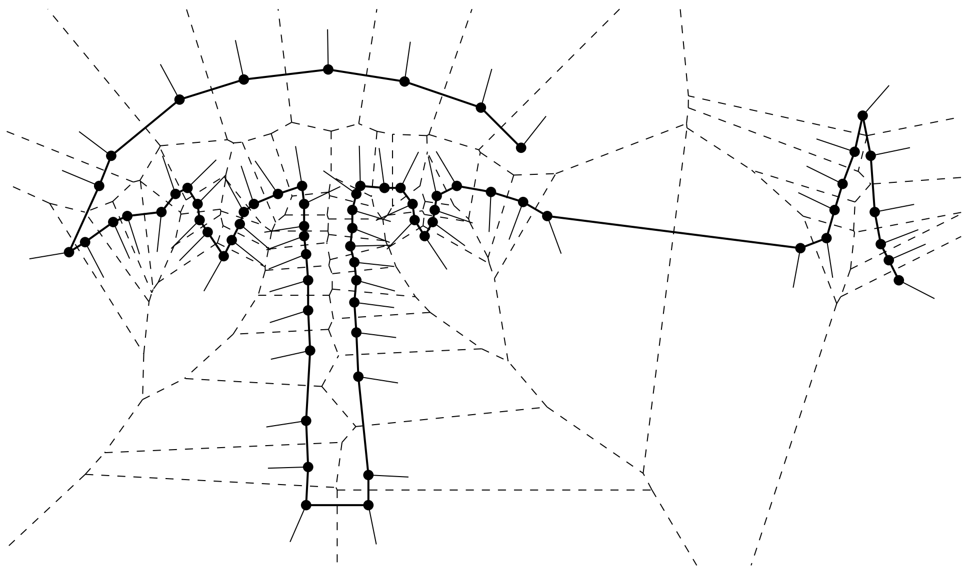
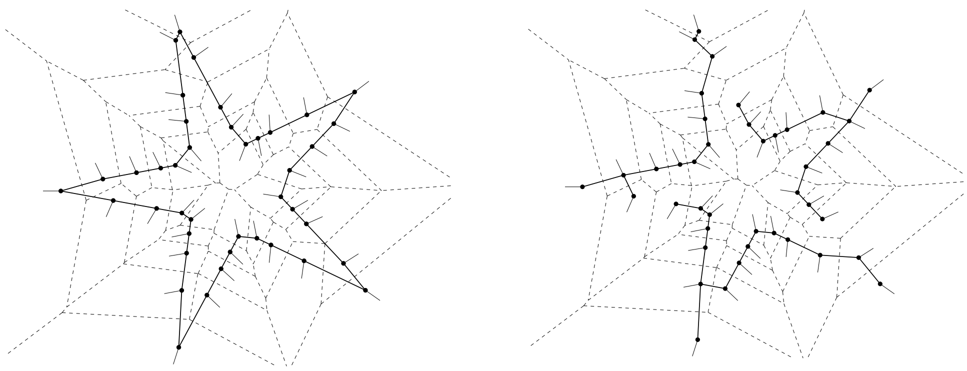
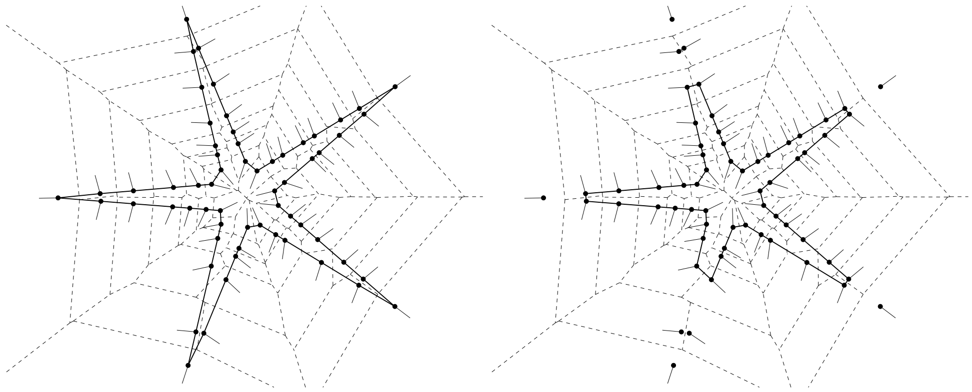


Fig. 43. Reconstruction of mushroom data set[8] after only 1 iteration.



GathanG reconstruction. Crust.

Figure 44: Voronoi diagram, pole directions and GathanG and crust reconstruction of star.



Angle $\alpha = 10°$. Angle $\alpha = 20°$.

Figure 45: Voronoi diagram, pole directions and reconstruction of sharp star with different angle parameters.

from the literature as well as some new data sets. We set the angular lower bound α to $10°$ on all these runs. Figures 39 and 40 appeared in[13] and [2], respectively. The reconstruction in each was calculated in a single iteration. The sample points in Figure 41 were generated using Althaus and Mehlhorn's software described in [2]. The reconstruction was also calculated in a single iteration. Figure 42 appeared in [8] and required two iterations of the algorithm. After a single iteration there is an erroneous long edge connecting the endpoint of one of the curves to a corner of the other (Figure 43.) This erroneous edge is detected and avoided in the second iteration.

Figure 44 contains the output of GathanG and CRUST on a star data set of our own construction. Our algorithm completed in a single iteration. Note that while the star is fairly uniformly sampled, the nearest neighbors are often not adjacent in the correct reconstruction. Figure 45 contains a sharper star data set, again of our construction. With the angular lower bound α set to $10°$, GathanG correctly reconstructs the curve in a single iteration. However, if α is set to $20°$, then it is no longer a lower bound on the corner angles, and GathanG truncates the arms of the star.

10. Conclusion

We presented an $O(n \log n)$ algorithm which guarantees reconstruction of a family of closed curves under appropriate sampling conditions. We also described an implementation of this algorithm which performs well even when sampling conditions are violated or the curves are open. Our algorithm is asymptotically faster than Ramos and Funke's algorithm in [13] and it does not require a bound on the angle $\angle(p_1, q, p_2)$ between q and the adjacent sample points p_1 and p_2. On the other hand, Ramos and Funke give reconstruction guarantees even when the curves are open and we do not.

Our algorithm is also asymptotically faster than the linear programming algorithm by Althaus and Mehlhorn [1]. Their algorithm cannot handle open curves.

Motivation for our work in 2d curve reconstruction was to generate ideas for 3d surface reconstruction. Nearest neighbors, the ratio test and pole directions are used in 3d surface reconstruction algorithm described in [7].

11. Acknowledgement

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