Visibility with multiple diffuse reflections

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Abstract

The combinatorial complexity of the region lit up from a point light source inside a simple n-gon with perfectly reflecting edges, after at most k *specular* reflections was established as $O(n^{2k})$ [2]. A lower bound of $\Omega((n/k)^{2k})$ was also established, matching the upper bound for any fixed k. In a real situation, surfaces may not be perfect mirrors; indeed most surfaces may be non-shiny or rough, causing *diffuse* reflection, rather than specular reflection. In contrast to the result of [2], the combinatorial complexity of the region lit up from a point inside a simple n-gon after k diffuse reflections is established here to be $O(n^{2\lceil (k+1)/2 \rceil +1})$.

Keywords: visibility, diffuse reflection, combinatorial complexity.

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1 Introduction

In recent years, visibility problems have been studied extensively (see [12, 14]). Visibility computation in computer graphics, motion planning, robotics and computer vision have been the prime source of many problems in this area. Two points inside a simple polygon are mutually *visible* if the line segment joining them is not obstructed by any edge of the polygon. Several algorithms for computing the region visible from a point light source inside a simple polygon exist [5, 9, 7]. The computation of the region of a simple polygon which is weakly visible from an internal segment [10, 7], or a convex set [6], are also well studied. Certain portions of the polygon that are not directly illuminated from the source may become visible due to one or more reflections on the bounding edges of the polygon. Reflection at a point is called *specular* if the reflected ray is directed as per the standard law of reflection; the angle of incidence is the same as the angle of reflection. Most reflecting surfaces cause another type of reflection, called *diffuse* reflection, where light incident at a point is reflected in all possible interior directions. In this paper, we study properties of the region visible from a point light source inside a simple polygon due to multiple *diffuse* reflections on the edges of the polygon. We assume that light incident at a vertex is absorbed and not reflected.

The combinatorial complexity of visibility polygons due to reflection was first studied by Aronov *et al.* [1]. A tight $\Theta(n^2)$ worst-case bound was established for visibility polygons when at most one diffuse reflection is permitted. They also studied the combinatorial complexity of the region visible due to at most one specular reflection. A tight $\Theta(n^2)$ bound was also established for at most one specular reflection in [1]. In a later paper [2], they investigated the case where at most k specular reflections are permitted. They produced an $O(n^{2k})$ upper bound and a matching $\Omega((n/k)^{2k})$ worst-case lower bound when at most a constant number of k specular reflections are permitted. In contrast to the result of [2], we show in this paper that the upper bound improves to $O(n^{2\lceil (k+1)/2\rceil+1})$, when specular reflections are replaced by diffuse reflections. In establishing this non-trivial bound, we reveal several properties of diffuse reflection that have not been studied erstwhile.

Among the different variants of visibility, *link visibility* [15, 16, 14] is probably most closely related to visibility with reflection. Two points inside a polygon are said to be L_k*-visible* if there is a polygonal path (called a *link path*) of k edges, connecting the two points, where the path remains inside the polygon or at most touches its boundary. Link visibility has been studied extensively (see [15, 16, 11, 4, 8]). If the link path of k edges is restricted to bend only at points other than vertices on the boundary of the polygon, then the two points joined by the path are clearly mutually visible due to $k-1$ diffuse reflections. However, there are examples where two points inside a polygon are L_k-visible for some $k \geq 1$ but not mutually visible after $k-1$ diffuse reflections [13].

The remainder of this paper is organized as follows. In section 2 we present the notation that will be used throughout the paper, and discuss some preliminaries. Section 3 establishes the upper bound on the combinatorial complexity of the region visible due to at most k diffuse reflections. Section 4 describes an algorithm for computing the visible region. Finally, we conclude in section 5.

2 Preliminaries

Let P= $(v_1, v_2, ..., v_n)$ be a simple polygon with no three collinear vertices. Let $int(X)$ and $bd(X)$ denote the relative interior and boundary of a region X, respectively. If p and q are two points on $bd(P)$, let $bd(p,q)$ denote the part of $bd(P)$ traversed from p to q in counterclockwise order.

Two points in P are *mutually visible* if the interior of the line segment joining them lies in $int(P)$. A point p is *weakly visible* from an edge or an internal line segment st, if there is a point z in the *interior* of st such that p and z are mutually visible. A point y is *visible under*

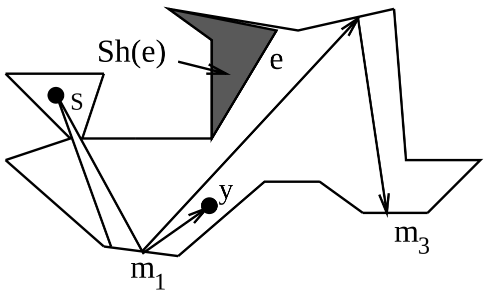

Figure 2.1: Definition of k-visibility

diffuse reflection from a point x, if there exists a point p lying in the interior of an edge of P, such that p is mutually visible from both x and y.

Let S denote a point light source inside P. A point y is 1-visible from S if y and S are mutually visible. For $k > 1$, a point $y(= p_k)$ is *k-visible* from a point $S(= p_0)$, if there exists points $p_1, p_2, ..., p_{k-1}$ in the interiors of edges of P, such that p_i and p_{i+1} are mutually visible for $0 \leq i \leq k-1$. We say that the ray emitted from S reaches y after $k-1$ *diffuse reflections* at points $p_1, p_2, ..., p_{k-1}$. In Figure 2.1, y is *2-visible* from S; y is visible from S after one diffuse reflection. We assume that the light incident at a vertex is absorbed and not reflected further.

Let $V(S)$ denote the portion of P visible from S. We know that $V(S)$ has $O(n)$ edges and at most one edge of $V(S)$ lies on each edge of P [5, 7]. For $k \geq 0$, let $Vd_k(S)$ denote the polygonal region consisting of points that are l-visible from S, for some $1 \leq l \leq k+1$. Informally, $Vd_k(S)$ is the set of points that receive light from S after at most k diffuse reflections. We have $Vd_0(S) = V(S)$.

We define a *mirror at the kth stage of reflection* inductively. The edges or connected subsets of edges of $bd(P)$ that occur on the boundary of $V(S)$ constitute the mirrors at the first stage of reflection. Let m be a mirror at the $(k-1)$st stage of reflection, and $V(m) \subseteq P$ be the set of points weakly visible from the mirror m. We know that $V(m)$ has at most one edge lying on each edge of P [7]. We call $V(m)$ the *visibility polygon* of m. An edge of $V(m)$ lying on $bd(P)$ is a mirror at the kth stage of reflection. In Figure 2.1, m_3 is a mirror at the third stage of reflection.

Edges of $V(m)$ that lie in the interior of P are called *shadow edges*. A shadow edge connects a reflex vertex of P and a point on $bd(P)$. In Figure 2.1, e is a shadow edge of $V(m_1)$. Let $e = rq$ be a shadow edge of $V(m)$ where r is a reflex vertex of P and q the other endpoint on $bd(P)$. The ray starting at q and passing through r, must also pass through a point of m before exiting the polygon P (see [7]). P is divided by e into two sub-polygons with disjoint interiors. The interior of one of these two sub-polygons does not intersect $V(m)$. This sub-polygon, denoted $Sh(e)$, is called the *shadow* of e. A shadow edge of a visibility polygon is a *right shadow edge* (*left shadow edge*), if its shadow falls on its right (left) when the shadow edge is traversed in the direction of light emanating from the mirror. Let $f \neq e$ be another shadow edge of $V(m)$. Since f is a boundary edge of $V(m)$, and the shadow $Sh(e)$ does not intersect $V(m)$, f can not intersect $Sh(e)$ [1].

The union of mirrors at the kth stage of reflection, lying on an edge e of P, form connected components. Each point in a connected component is weakly visible from some mirror at the $(k-1)$st stage of reflection. Such a connected component is called a *reflecting segment at the kth stage* of reflection. Sometimes we use the phrase *reflecting segment*, where the stage of reflection is obvious from the context. The region $V(c)$, weakly visible from a reflecting segment, is called its *visibility polygon*.

3 Upper bound on the combinatorial complexity of $Vd_k(S)$

The mirrors at the first stage of reflection are created on the edges that are directly lit up by the source S. Since the number of edges of $V(S)$ is $O(n)$ (see [5, 7]), the number of mirrors that cause the first stage of reflection is n. Each of these n mirrors creates at most one reflecting segment on each edge of P [7]. Therefore, there are $O(n^2)$ mirrors involved in the second stage of reflection. Indeed, this worst case can be realized as illustrated in Figure 3.2, where $\Omega(n)$ mirrors are created on each of $\Omega(n)$ edges, thus creating $\Omega(n^2)$ mirrors for the second stage of reflection (see [1]). This bound can be generalized for multiple reflections; we observe that each mirror at a certain stage of reflection can create at most n mirrors for the next stage of reflection. So, an obvious upper bound on the number of mirrors at the kth stage of reflection, is $O(n^k)$. Let $m_1, m_2, ..., m_t$ be the set of all mirrors up to the kth stage of reflection in any order, and $m_0 = S$. Clearly, $Vd_k(S) = \cup_{i=0}^{t} V(m_i)$, a union of $(t+1) = O(n^k)$ visibility polygons. Alternatively, $Vd_k(S)$ may be viewed as the union $\cup_{i=0}^{a} V(c_i)$, where $c_1, c_2, \ldots, c_a$ are reflecting segments over all the k stages of reflection and $c_0 = S$. The major contribution of this paper is to show that the number of reflecting segments at the kth stage of reflection is $O(n^{\lceil (k+1)/2 \rceil})$. $Vd_k(S)$ is shown to be the union of $O(n^{\lceil (k+1)/2 \rceil})$ visibility polygons, leading to

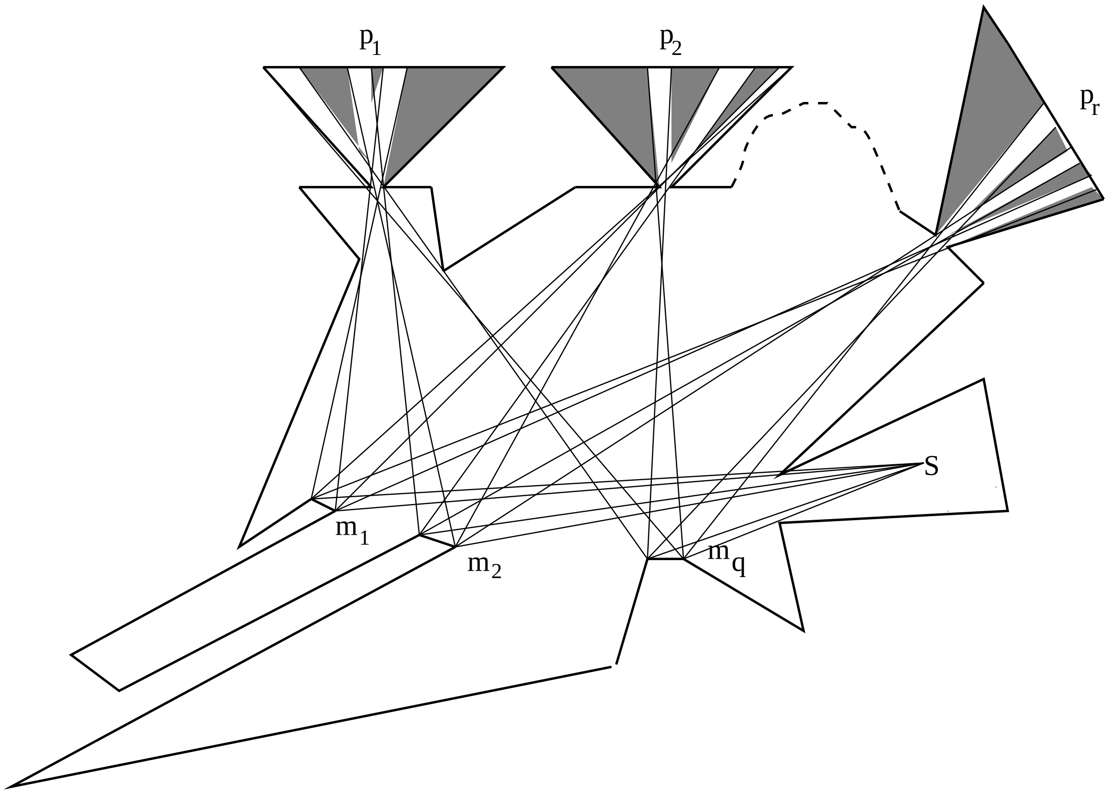

Figure 3.2: $Vd_1(S)$ has $\Omega(n^2)$ edges

Figure 3.3: Three disjoint reflecting segments on e_i

an $O(n^{2\lceil (k+1)/2 \rceil +1})$ bound on the number of vertices of $Vd_k(S)$.

We wish to establish an upper bound on the number of reflecting segments at the $(l+1)$st stage of reflection. Consider any edge $f = u_1u_2$ of the polygon P. Here u_2 is the first vertex in a counterclockwise traversal of $bd(P)$ starting at u_1, or equivalently $u_1u_2 = bd(u_1, u_2)$. In the rest of the paper we follow this convention while writing a segment on $bd(P)$. Suppose f has a reflecting segment at the $(l-1)$st stage of reflection. Then, the following lemma implies that the entire edge f is a reflecting segment at the $(l+1)$st stage of reflection.

LEMMA 3.1 Let e be an edge of P that has a reflecting segment at the lth stage of reflection. Then, the entire edge e has is a reflecting segment at the $(l+2)$nd stage of reflection.

PROOF. Let the *extension* of e be the maximal connected segment in P that contains e. The extension of e is an open set and the end points of its closure lie on $bd(P)$. The lth reflection from the edge e illuminates points of $bd(P)$ in the neighbourhoods of these endpoints. The $(l+1)$st reflections from these neighbourhoods flood the entire edge e with light. $\square$

So, to consider only non-trivial cases, we assume that f does not contain any reflecting segment at the $(l-1)$st stage of reflection. Let $E = \{e_i = v_{i1}v_{i2} | 1 \le i \le h\}$ be the sequence of edges containing mirrors at the lth stage of reflection that reflect light on f, where the edges $e_1, e_2, ..., e_h$ appear in counterclockwise order when $bd(P)$ is traversed starting at u_1. Every mirror lying on an edge $e_i \in E$ may not reflect light on f. Only those mirrors that lie on edges in E and create reflecting segments on f are of interest to us. Such mirrors are called *contributing* mirrors for f. Note that these are mirrors at the lth stage of reflection. Consider the union of contributing mirrors on any edge $e_i \in E$. We call each connected component of this union a *contributing reflecting segment*. Note that parts of e_i outside this union may also be visible from mirrors of the $(l-1)$st stage of reflection. However, we ignore these visible portions since they do not illuminate any part of f. In Figure 3.3, the contributing reflecting segment c_2 consists of two overlapping mirrors, created by two reflecting segments at the $(l-1)$st stage of reflection. The segment c_3 consists of three mirrors created by three reflecting segments at the $(l-1)$st stage of reflection. Let Π_{e_i} denote the set of contributing reflecting segments on e_i and let $\Pi_E = \{\cup_{e_i \in E} \Pi_{e_i}\}$. For each such contributing reflecting segment c, consider its visibility polygon $V(c) \subseteq P$. We know that $V(c)$ has at most one edge lying on the edge f. Thus a contributing reflecting segment c can contribute to exactly one reflecting segment at the $(l+1)$st stage of refelction, on the edge f. Due to this property, the number of such reflecting segments on f is upper bounded by the cardinality of Π_E. Hence we count the number of elements in Π_E.

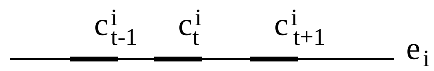

Figure 3.4: c^i_t is flanked by reflecting segments

Let $c^i_1, c^i_2 \ldots, c^i_q$ be the contributing reflecting segments on an edge $e_i \in E$. For every edge e_i we eliminate the first segment c^i_1 and the last segment c^i_q from Π_E. There are at most $2n$ eliminated segments, over all edges e_i. Let Π'_E denote the remaining reflecting segments in Π_E. Let c be a reflecting segment at the $(l-1)$st stage of reflection that reflects light to $s \in \Pi'_E$. We charge c for the contribution of s to f. We show in the following lemma that each reflecting segment at the $(l-1)$st stage of reflection is charged at most once by this charging scheme. Consider $c^i_t \in \Pi'_E$ on edge e_i. Due to our elimination criterion, there must be two segments c^i_{t-1} and c^i_{t+1} flanking c^i_t on e_i. (see Figure 3.4). Let m_1, m_2 and m_3 be the three reflecting segments at the $(l-1)$st stage of reflection, that reflect light into c^i_{t-1}, c^i_t and c^i_{t+1}, respectively.

LEMMA 3.2 The reflecting segment m_2 cannot reflect light to more than one segment in Π'_E.

PROOF. Let $s^i_1 \subseteq c^i_{t-1}, s^i_2 \subseteq c^i_t$ and $s^i_3 \subseteq c^i_{t+1}$ be three segments created by m_1, m_2 and m_3 on e_i (see Figure 3.5). For the sake of contradiction suppose m_2 also reflects light to a component $c^j_p \in \Pi'_E$ on edge e_j. We know that m_2 can reflect light into at most one segment in any edge of P. So, $j \neq i$. Since the extreme segments on each edge are eliminated in the scheme above, c^j_p is neither the first nor the last reflecting segment on e_j. Let $c^j_{p-1} \in \Pi'_E$ and $c^j_{p+1} \in \Pi'_E$ be two segments flanking c^j_p on e_j. Let m'_1 and m'_3 be the reflecting segments at the $(l-1)$st stage of reflection, reflecting light into c^j_{p-1} and c^j_{p+1}, respectively. Let $s^j_1 \subseteq c^j_{p-1}, s^j_2 \subseteq c^j_p$ and $s^j_3 \subseteq c^j_{p+1}$ be the segments created by m'_1, m_2 and m'_3, respectively (see Figure 3.5). Since Π'_E is made of only contributing reflecting segments, all the segments in the set $\{s^a_1, s^a_2, s^a_3, a \in \{i, j\}\}$, reflect light into $f = u_1u_2$. We assume without loss of generality that e_j appears before e_i in a counterclockwise traversal starting at u_1 (see Figure 3.5). Let $e_i = v_{i1}v_{i2}$, $e_j = v_{j1}v_{j2}$ and $m_2 = x_1x_2$. Since f does not have any reflecting segment at the $(l-1)$st stage of reflection, m_2 can not lie on f. We have the following cases.
Case 1. $m_2 \subseteq bd(v_{i2}, u_1)$
Case 2. $m_2 \subseteq bd(u_2, v_{j1})$
Case 3. $m_2 \subseteq bd(v_{j2}, v_{i1})$

Case 1: Consider the edge e_i. The segment s^i_2 is an edge of $V(m_2)$, and indeed, the only edge of $V(m_2)$ lying on e_i. The segments s^i_2 and s^i_3 lie in disjoint reflecting segments c^i_t and c^i_{t+1}, respectively. The segment s^i_3 lies inside a right shadow of $V(m_2)$. Let rq be the shadow edge such that r is the reflex vertex and q is the other vertex. The point q lies on $bd(P)$ between s^i_2 and s^i_3. Since rq is a right shadow edge, r must lie in $bd(x_2, v_{i1})$. If r lies in $bd(x_2, v_{j1})$ then e_j lies entirely in the shadow, contradicting the assumption that $V(m_2)$ has an edge s^j_2 lying on e_j. So, r must lie in $bd(v_{j2}, v_{i1})$. Recall that the shadow edge when extended backwards must pass through a point of the mirror m_2 before exiting the polygon. So, a point in m_2, the point $r \in bd(v_{j2}, v_{i1})$, and the point q in e_i must be collinear. Since s^i_3 reflects light to f, there

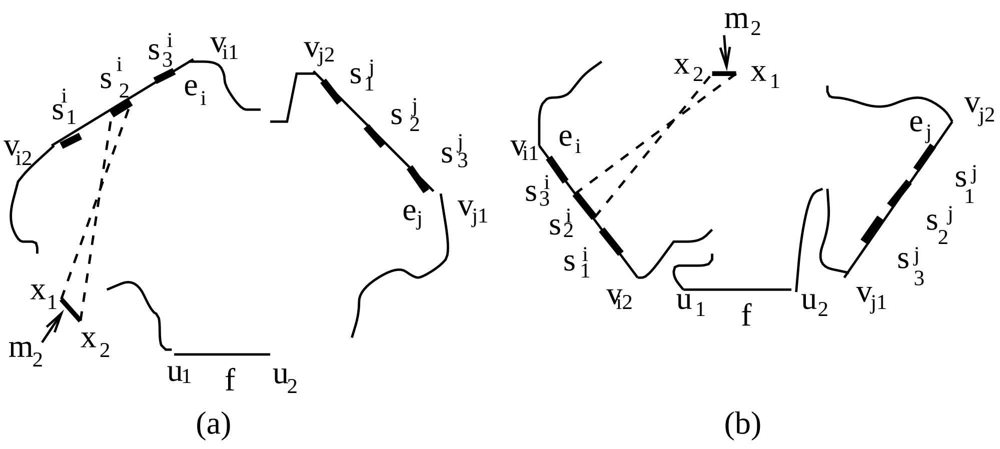

Figure 3.5: (a) m_2 appears in $bd(v_{i2}, u_1)$ (b) m_2 appears in $bd(v_{j2}, v_{i1})$

must be points s and t on s_3^i and f, respectively, so that s and t are mutually visible. This leads to a contradiction because q and m_2 lie on one side of st and r lies on the other side.

Case 2 is symmetrical to Case 1.

Case 3: As in Case 1, The segment s_2^i is an edge of $V(m_2)$, and indeed, the only edge of $V(m_2)$ lying on e_i. The mirrors s_1^i and s_2^i lie in disjoint reflecting segments c_{t-1}^i and c_t^i, respectively. So, s_1^i must be in a left shadow of $V(m_2)$. Let rq be the shadow edge such that r is a reflex vertex and q is a point of $bd(P)$ between s_1^i and s_2^i. Since rq is a left shadow edge, r must lie in $bd(v_{i2}, x_1)$. If r lies in $bd(v_{j2}, x_1)$, then the shadow encloses the entire edge e_j, contradicting the assumption that s_2^j is an edge of $V(m_2)$. Now consider the possibility that r lies in $bd(v_{i2}, u_1)$. Since s_1^i reflects light to f, there must be two points s and t in s_1^i and f, respectively, so that s and t are mutually visible. Being a shadow edge rq must be collinear with some point in m_2. A contradiction results because m_2 and q are on one side of st and $r \in bd(v_{i2}, u_1)$ is on the other side of st. Using similar arguments we can show that r can not lie in $bd(u_2, v_{j1})$ because s_3^j reflects light to f. $\square$

Lemma 3.3 Any edge f of P has at most $g(n,l) = 3n^{\lceil l/2 \rceil} + 2\Sigma_{i=1}^{\lceil l/2 \rceil - 1} n^i$ reflecting segments at the $(l+1)$st stage of reflection.

Proof. Let $H(l)$ be: *Any edge of P has at most $3n^{\lceil l/2 \rceil} + 2\Sigma_{i=1}^{\lceil l/2 \rceil - 1} n^i$ reflecting segments at the $(l+1)$st stage of reflection.* We show by induction that $H(l)$ is true for all $l \geq 0$. The base case $l = 0$ corresponds to direct visibility from the source. $Vd_0(S) = V(S)$ has at most one edge on every edge of P. Since there are at most n mirrors participating in the first reflection, there can be at most n edges of $Vd_1(S)$ lying on any edge of P. So, our induction hypothesis holds for $0 \leq l \leq 1$. Suppose $H(l-2)$ is true for $l \geq 2$. Then, using Lemma 3.2, we show that $H(l)$ also holds. Since $H(l-2)$ holds, there are at most $3n^{\lceil (l-2)/2 \rceil} + 2\Sigma_{i=1}^{\lceil (l-2)/2 \rceil - 1} n^i$ reflecting segments at the $(l-1)$st stage of reflection on any edge of P. Considering all the n edges of

P, there are at most $3n^{\lceil (l-2)/2 \rceil +1} + 2n\Sigma_{i=1}^{\lceil (l-2)/2 \rceil -1} n^i$ reflecting segments at the $(l-1)$st stage of reflection. Since each such reflecting segment at the $(l-1)$st stage of reflection can contribute a segment only in a unique segment of Π'_E (see Lemma 3.2), the total number of segments in Π'_E is $3n^{\lceil (l-2)/2 \rceil +1} + 2n\Sigma_{i=1}^{\lceil (l-2)/2 \rceil -1} n^i$. The remaining segments of $\Pi_E \setminus \Pi'_E$ are $2n$. So, the cardinality of Π_E, and therefore, the total number of reflecting segments created on f is at most $3n^{\lceil (l-2)/2 \rceil +1} + 2n\Sigma_{i=1}^{\lceil (l-2)/2 \rceil -1} n^i + 2n$, which is same as the desired bound $3n^{\lceil l/2 \rceil} + 2\Sigma_{i=1}^{\lceil l/2 \rceil -1} n^i$. $\square$

In what follows, we show that $Vd_k(S)$ is the union of of $O(n^{\lceil (k+1)/2 \rceil})$ visibility polygons. The number of edges of $Vd_k(S)$ on $bd(P)$, and the combinatorial complexity of $Vd_k(S)$ easily follow from this bound.

LEMMA 3.4 $Vd_k(S)$ is the union of $O(n^{\lceil (k+1)/2 \rceil})$ visibility polygons.

PROOF. We know that $Vd_1(S)$ is the union of $O(n)$ visibility polygons including $V(S)$. $Vd_2(S)$ is the union of $V(S)$, $Vd_1(S)$, and the visibility polygons in $\{V(e) | \, e$ is a mirror at the second stage of reflection$\}$. Since $Vd_1(S)$ has $\Theta(n^2)$ edges [1], the number of such mirrors at the second stage of reflection is $O(n^2)$. So, $Vd_2(S)$ is the union of $O(n^2)$ visibility polygons. The lemma therefore holds for $k = 1$ and 2.

Now consider cases where $k \geq 3$. Edges of P that get lit up for the first time due to at most $k-3$ reflections, get fully lit up after the $(k-1)$st reflection (see Lemma 3.1). Each such edge e of P contributes the region $V(e)$ to $Vd_k(S)$ in the kth stage of reflection. There are at most n such visibility polygons, which are subsets of $Vd_k(S)$. The remaining edges of P have reflecting segments only at the $(k-1)$st and kth stages of reflection. Applying Lemma 3.3, it follows that $Vd_k(S)$ is the union of the visibility polygons of at most $n + n * g(n, k-2) + n * g(n, k-1) = O(n^{\lceil (k+1)/2 \rceil})$ reflecting segments.

$\square$

COROLLARY 3.5 The number of edges of $Vd_k(S)$ that lie on $bd(P)$ is $O(n^{\lceil (k+3)/2 \rceil})$.

PROOF. $Vd_k(S)$ is the union of $O(n^{\lceil (k+1)/2 \rceil})$ visibility polygons, each having $O(n)$ edges. $\square$

Corollary 3.5 gives a bound on the number of edges of $Vd_k(S)$ that lie on $bd(P)$. However, $Vd_k(S)$ also has edges and vertices lying inside P. In order to derive a bound on the combinatorial complexity of $Vd_k(S)$ we further need a few lemmas.

LEMMA 3.6 Let m be a segment lying on an edge of P. Any segment $s \subseteq int(P)$ can intersect at most two shadow edges of $V(m)$.

Proof. The segment s can not intersect any edge of $V(m)$ that lies on $bd(P)$. So, s must intersect a shadow edge. If s is a subset of a shadow edge then it clearly can not intersect another shadow edge because no two shadow edges of $V(m)$ intersect. Otherwise, s may intersect a shadow edge e at a single point p. The part of s that lies inside the shadow $Sh(e)$, does not intersect $V(m)$ or any other shadow edge of $V(m)$ because a shadow can not contain another shadow edge in its interior. The part of s that lies outside $Sh(e)$ may intersect another shadow edge f of $V(m)$ and enter the shadow $Sh(f)$. Using similar arguments we can show that the part of s that lies inside this shadow too can not intersect any other shadow edge of $V(m)$.

□

Lemma 3.7 Let c_1,c_2,..., c_a be reflecting segments on the edges of P. The number of vertices in the union of the visibility polygons $V(c_i), 1 \leq i \leq a$ is $O(a^2 n)$.

Proof. The total number of vertices of the union of these a visibility polygons include the vertices of these polygons and the vertices created by the intersection of these polygons. The total number of edges in these visibility polygons is $O(an)$. Now consider the intersections between edges of the visibility polygons that lie inside P. By Lemma 3.6, we know that each of these $O(an)$ edges can intersect each of the a visibility polygons at most twice, giving a total of $O(a^2 n)$ intersections. □

Theorem 3.8 $Vd_k(S)$ has $O(n^{2\lceil (k+1)/2 \rceil + 1})$ vertices.

Proof. Follows from Lemmas 3.4 and 3.7. □

4 Algorithm for computing $Vd_k(S)$

We first compute the visibility polygon $V(S)$ by any linear time algorithm as in [5, 7]. Edges of $V(S)$, lying on $bd(P)$ will act as mirrors in the first stage of reflection. Suppose we have computed the reflecting segments at the $(i-1)$st stage of reflection, where $i \geq 2$. We know that there are $O(n^{\lceil i/2 \rceil})$ reflecting segments at the $(i-1)$st stage of reflection over all edges of P (see Lemma 3.3). For each reflecting segment c at the $(i-1)$st stage of reflection, we compute the visibility polygon $V(c)$, and determine the reflecting segments at the ith stage of reflection as follows. Consider the set R of the segments in the intersection of $V(c)$ and $bd(P)$, computed over all reflecting segments c at the $(i-1)$st stage of reflection. There are $O(n^{\lceil (i+2)/2 \rceil})$ segments in R. The reflecting segments at the ith stage of reflection are the connected components of the union of all the segments in R. Computing the union requires sorting the endpoints of the segments in R within each edge of P. Since there are $O(n^{\lceil (i+2)/2 \rceil})$ such endpoints, the time required for this computation is $O(n^{\lceil (i+2)/2 \rceil} \log n^{\lceil (i+2)/2 \rceil})$. Thus, the computation of the reflecting segments over all the k stages of reflection takes $O(\Sigma_{i=1}^{k} n^{\lceil (i+2)/2 \rceil} \log n^{\lceil (i+2)/2 \rceil})=$ $O(k * n^{\lceil (k+2)/2 \rceil + 1} \log n)$ time.

Consider edges of P that have reflecting segments at the $(k-2)$nd stage of reflection. Each such edge acts as a single reflecting segment at the kth stage of reflection (see Lemma 3.1). The remaining edges of P have reflecting segments only at the $(k-1)$st and the kth stages of reflection. Let $c_1, c_2, \ldots, c_a$ be all these reflecting segments. We know that $Vd_k(S)$ is the union of the visibility polygons in the set $Q = \{V(c_i) | 0 \leq i \leq a\}$, where $c_0 = S$. In Lemma 3.4 we have shown that $a = O(n^{\lceil (k+1)/2 \rceil})$. For each reflecting segment c_i, the visibility polygon $V(c_i)$ can be computed in linear time by the algorithm of [7]. After computing all these visibility polygons, we apply a divide-and-conquer technique to compute $Vd_k(S)$ as the union of the polygons in Q.

We partition Q into two roughly equal subsets Q' and Q''. The union of the polygons in Q' (Q'') is computed recursively. Let P' and P'' denote the union of the polygons in Q' and Q'', respectively. Finally, we compute $Vd_k(S)$, the union of P' and P'', as follows. Use the algorithm of Bentley and Ottmann [3], to compute the intersections between the edges of P' and P''. The points of intersections betweeen edges occuring in the interior of P, are vertices of $Vd_k(S)$; exactly one of the four quadrants in the neighbourhood of such an intersection point is outside $Vd_k(S)$. The intersection between edges that lie on $bd(P)$ can also be easily computed. Finally, we do a routine traversal of the edges of P' and P'' to get the boundary of $Vd_k(S)$.

Since each visibility polygon has $O(n)$ edges, there are a total of $O(an)$ edges in the a visibility polygons creating P' and P''. By Lemma 3.7, the union of all the a visibility polygons from Q' and Q'' has $O(a^2 n)$ vertices. Thus the segment intersection algorithm takes $O(a^2 n \log na + na \log na)$ time. The time $T(a)$ for this divide-and-conquer strategy can be expressed as

$$T(a) = \begin{cases} 2T(a/2) + O(a^2 n \log na) & \text{for } a > 1 \\ O(n) & \text{for } a = 1. \end{cases}$$

which solves to $O(a^2 n \log na)$.

Substituting $a = O(n^{\lceil (k+1)/2 \rceil})$, we have the following theorem.

THEOREM 4.1 $Vd_k(S)$ can be computed in $O(k * n^{2\lceil (k+1)/2 \rceil + 1} \log n)$ time.

5 Conclusion

Although we have established non-trivial bounds on the combinatorial complexity of the region of an n-gon, visible from a point light source due to $k \geq 1$ diffuse reflections, there is a huge gap between the lower and upper bounds. All our studies prompt us to believe that the real bound is $O(n^2)$.

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