Surface Reconstruction Simplified

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Abstract

The problem of computing a piecewise linear approximation to a surface from its sample has been a focus of research in solid modeling and graphics due to its widespread applications in computer aided design. A typical input for this *surface reconstruction* problem consists of three dimensional coordinates of the sampled points. The crust algorithm of [1] reconstructs a surface with topological and geometric guarantees using the Voronoi diagram of the input point set. We present new observations that simplify both the algorithm and proofs of crust substantially. Our theory is supported by the output of our program on several data.

1 Introduction

A number of applications ranging over CAD, computer graphics and mathematical modeling involve the computation of a piecewise linear approximation to a surface from its sample points. In practice, the input point set is generated using a variety of tools such as laser range scanners, contact probe digitizers and medical imageries like CT or MRI scanners. In some cases, the input may be equipped with additional information. The contour data generated with CT or MRI scanning have the points lying on planar slices. Sometimes estimates of surface normals are also provided along with the sample points. Special-purpose algorithms have been designed to exploit these special properties; for example, see [7, 8]. Although these additional information are helpful, they are not always available. Hoppe et al. [16] pointed out the need for addressing the problem of surface reconstruction under a general setting, where no additional information is available other than the space co-ordinates of the sample points. The two dimensional version of the problem, namely the curve reconstruction in plane from sample co-ordinates has been well researched. A variety of approaches [2, 5, 9, 10, 12, 15, 17] are known to work with theoretical guarantees.

In three dimensions, very few algorithms are known that provide performance guarantees. Hoppe et. al [16] presented an algorithm based on *zero sets* of a signed distance function. A similar approach is taken by Curless and Levoy [8]. Edelsbrunner reports success for a commercial software developed under propriety rights [11]. Bajaj, Bernardini and Xu [6] use α-shapes of Edelsbrunner and Mücke [13] to reconstruct scalar fields. However, these results do not provide assurance that the output surface is topologically equivalent to the sampled surface. Topological equivalence is a prime requirement in modeling applications. Clearly, it is not possible to compute

a surface that is faithful to the topology and geometry of the original one unless the sampling is sufficiently dense.

Amenta and Bern [1] proposed that the sampling density be proportional to the *local feature size* of the surface and presented an algorithm based on Delaunay triangulation. This algorithm guarantees that the output is homeomorphic and geometrically close to the sampled surface given that S is a smooth 2-manifold without boundary. The algorithm uses two passes of Voronoi (or its dual Delaunay triangulation) diagram computations, and also uses two major postprocessing steps what they called *filtration by normals* and *trimming*. Since typically the input sample set is large in most applications, elimination of any major step in the algorithm has a perceptible impact on the running time of the algorithm with most of the modern machines used for the purpose. To this end, Amenta and Choi [4] proposed a single pass Delaunay algorithm for reconstruction recently. The algorithm is based on the following main observation. Let T be a set of triangles spanning all sample points, which satisfy three conditions, namely, (A) T contains all triangles whose dual Voronoi edges intersect S. (B) each triangle in T is small, that is, their circumcircle has a small radius compared to the local feature size, (C) all triangles in T are "flat", that is, their normals make small angles with the normals to the surface at their vertices. The algorithm of [4] detects the set of triangles in the Delaunay triangulation that satisfy condition (C) explicitly using the estimated normals at the samples from the Voronoi diagram. To satisfy condition (B) the algorithm checks the condition: (D) if the dual Voronoi edge of a triangle t has a point close to the plane of t. They argue that condition C and D together imply the condition (A) and (B).

We make new observations that further simplify the algorithm for computing a set T satisfying the three conditions, A, B and C. We show that the condition (B) implies condition (C). Thus, one needs not check the condition (C) explicitly. Also, we show that it is quite simple to detect triangles that satisfy condition (A) and (B). For this, we simply check the following. Let p be a sample point and e be a Voronoi edge in the Voronoi cell of p. We determine if e has a point x, where px makes an angle close to $\pi/2$ with the estimated normal at the sample point p. If this condition is satisfied for all Voronoi cells adjacent to e, we include its dual in our desired set T. This crucial observation not only provides a simple algorithm, but also makes the proofs simpler as compared to [1, 4]. Another fallout of this simplicity is that we arrive at better constants involved in the proofs. Figure 1 shows the schematic comparison between our algorithm and that of [4].

The paper is organized as follows. Section 2 contains definitions and the algorithm. Section 3 contains the proofs of all three conditions (A), (B) and (C). We describe our implementation experience in Section 4 and conclude in Section 5.

Figure 1: The left figure shows the schematic diagram of the logic behind Amenta-Choi algorithm, the right figure shows ours.

2 Definitions

We assume that the sampled surface S is a smooth manifold without boundary, embedded in R^3. We adopt the following definition of sampling density from [1].

2.1 ϵ-sampling

The *medial axis* of a surface S in R^3 is the closure of all centers of the spheres touching the surface in two or more points. The *local feature size*, $f(p)$ at a point p on S is the least distance of p to the medial axis. See Figure 2 for an illustration in two dimensions. A rough sketch of the medial axis of a curve is shown. A very useful property of $f(\cdot)$ is that it is 1-Lipschitz, that is, $f(p) \leq f(q) + |pq|$ for any two points p, q on S. A point set P is called an ϵ-sample of a surface S if every point $p \in S$ has a sample within a distance of $\epsilon f(p)$. We call ϵ the *sampling parameter* for P.

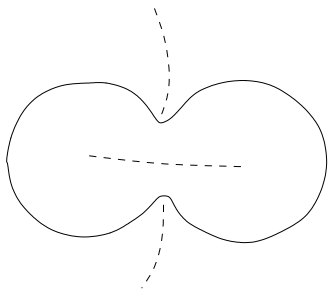

Figure 2: Medial axis shown with dotted curve.

2.2 Restricted Delaunay triangulation

The Delaunay triangulation of a point set $P \subseteq \mathbf{R}^3$, denoted D_P, is a unique triangulation of its convex hull which ensures that the circumscribing sphere of each tetrahedron in the triangulation is empty of any other input point. Of course, this is true for point sets in general positions which we assume for the simplicity of our exposition. The dual of the Delaunay triangulation is the Voronoi diagram V_P defined as the collection of convex cells V_p for each point $p \in P$ which tessellate $\mathbf{R}^3$. The cell V_p is the set of all points in $\mathbf{R}^3$ that have no points of P closer to them than p. The Delaunay triangulation has an edge pq if and only if V_p, V_q share a face, has a triangle pqr if and only if V_p, V_q and V_r share an edge, and a tetrahedron $pqrs$ if and only if V_p, V_q, V_r and V_s share a Voronoi vertex.

A surface $S \supseteq P$ is tessellated with the Voronoi cells. Define the *restricted Voronoi diagram* $V_{P,S}$ as the collection of *restricted Voronoi cells* $V_{p,S} = V_p \cap S$. The dual of these restricted Voronoi cells defines the *restricted Delaunay triangulation* $D_{P,S}$. Specifically, an edge pq is in $D_{P,S}$ if and only if $V_{p,S} \cap V_{q,S}$ is nonempty; a triangle pqr is in $D_{P,S}$ if and only if $V_{p,S} \cap V_{q,S} \cap V_{r,S}$ is nonempty. Assuming that S does not pass through a Voronoi vertex in general position we do not have any tetrahedron in $D_{P,S}$. A result of Edelsbrunner and Shah [14] shows that the underlying space of $D_{P,S}$ is homeomorphic to S if the following *closed ball property* holds. Each $V_{p,S}$ is a topological closed disk, each nonempty pairwise intersection $V_{p,S} \cap V_{q,S}$ is a topological 1-ball, and each nonempty triplet-wise intersection $V_{p,S} \cap V_{q,S} \cap V_{r,S}$ is a single point, that is, a 0-ball. Amenta

and Bern used this result to show that if P is a sample of S with the sampling parameter less than 0.1, then $V_{P,S}$ satisfies the closed ball property and hence D_P contains the set of triangles of $D_{P,S}$ whose underlying space is homeomorphic to S.

2.3 Algorithm

Our goal is to determine a set of triangles that satisfy the following three conditions.

CONDITIONS
(A) The set of triangles include restricted Delaunay triangles.
(B) The circumcircle of each triangle t is small, that is, its radius is $O(\epsilon)f(p)$, where p is a vertex of t.
(C) The normal to t makes small angle $O(\epsilon)$ with the surface normal at the vertex p, where p contains the largest face angle inside t.

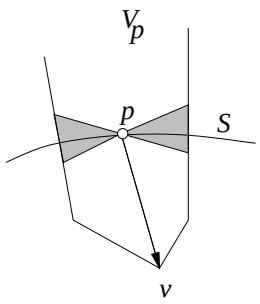

Figure 3: The shaded region contains all rays that make an angle in the range I with $\vec{v}$.

Let e be a Voronoi edge adjacent to a Voronoi cell generated by the sample p. The key to our reconstruction procedure is to determine if e has a point x so that the ray $p\vec{x}$ makes an angle close to $\pi/2$ with the normal at p. The normal to S at each sample point is estimated using the concept of "pole" as introduced in [1]. For each Voronoi cell V_p, the Voronoi vertex that is farthest from the sample point p is taken as the pole, and the line through p and the pole is almost normal to S. Notice that we do not need two poles as required in [1]. We will argue that the above simple algorithm computes a set of triangles T that satisfies A and B. Then we show that condition B implies the condition C. Let n_p denote the line normal to S at p and $v \in V_p$ be the farthest Voronoi vertex from p in V_p. Denote any ray from p to a point $y \in V_p$ as $\vec{y}$. For each $p \in P$, the procedure COMPUTET determines the set of Voronoi edges in V_p that has a point x where $\vec{x}$ makes an angle in the range $I = [\pi/2 - \theta, \pi/2 + \theta]$ with $\vec{v}$, where $\theta < 4\sin^{-1}\frac{\epsilon}{1-\epsilon}$. For an illustration in two dimensions, see Figure 3. Let e be an edge in the Voronoi cell V_p, and w_1, w_2 be its two endpoints. We compute $\angle\vec{w}_1\vec{v}$ and $\angle\vec{w}_2\vec{v}$ and check if the range of angles determined by these two angles intersects the desired range I. If it does, we mark e. We include e in E if it is marked for all three Voronoi cells adjacent with e. For implementation we can assume a small value for θ, say $\pi/8$.

COMPUTET(P)
    for each point $p \in P$ do
        Compute $\vec{v}$; $E := \emptyset$;

```
        for each Voronoi edge e ∈ V_p do
            Let w_1, w_2 be endpoints of e;
            Let θ_1 = min{∠w⃗_1 v⃗, ∠w⃗_2 v⃗} and θ_2 = max{∠w⃗_1 v⃗, ∠w⃗_2 v⃗};
            if [θ_1, θ_2] ∩ I ≠ ∅ mark e;
              if e is marked thrice E := E ∪ e;
            endif
        endfor
    endfor
    return the dual of E;
end
```

Let T be the set of dual Delaunay triangles of E. We show that T satisfies the three conditions A, B and C. Given such a set of triangles, the last step of the crust algorithm [1] first deletes all the triangles incident on *sharp edges*. An edge is called *sharp* if the angle between the two consecutive triangles around the edge is more than $3\pi/2$. An edge with a single incident triangle is also sharp. Deleting such triangles are safe since the restricted Delaunay triangles are not incident with sharp edges and they are included in the set. Next, a depth-first walk over the adjacency graph of the remaining triangles extracts the outer boundary of the underlying space of them. Amenta and Choi [4] prove that this boundary is homeomorphic to the sampled surface S under the assumption of three conditions A, B and C.

```
SURFACE(P)
    T :=COMPUTET(P);
    Delete any t ∈ T incident with a sharp edge;
    Output the outer boundary of the
            underlying space of T;
end
```

Theorem 1 *The algorithm* SURFACE *computes a piecewise linear surface homeomorphic to a surface for which P is an ϵ-sample with $\epsilon < 0.1$.*

We remark that the condition $\epsilon < 0.1$ comes from the fact that the restricted Delaunay triangulation is homeomorphic to S for this range of ϵ as shown in [1].

3 Conditions

We need the following two lemmas from [1] that says that the pole estimates the normal at the sample point.

Lemma 2 (**Amenta and Bern [1]**) *Let y be any point in V_p so that $|py| \geq \delta f(p)$ for $\delta > 0$. The acute angle between $\vec{yj}$ and n_p is less than* $\sin^{-1} \frac{\epsilon}{\delta(1-\epsilon)} + \sin^{-1} \frac{\epsilon}{1-\epsilon}$.

Using Lemma 2 it is shown in [1] that $\vec{v}$ approximates the normal direction n_p quite well.

Lemma 3 (**Amenta, Bern [1]**) *The acute angle between n_p and $\vec{v}$ is less than* $2 \sin^{-1} \frac{\epsilon}{1-\epsilon}$.

3.1 Condition A

Condition A requires that the restricted Delaunay triangles are in T. First, we need a technical lemma about local flatness of S.

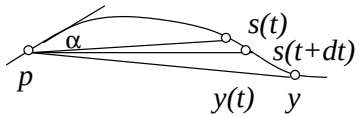

Figure 4: S is locally flat.

Lemma 4 *Let y be any point in $V_{p,S}$. The acute angle between n_p and $\vec{y}$ is larger than $\pi/2 - \frac{\epsilon}{1-4\epsilon}$.*

Proof. Let us parameterize py by length. Let $y(t)$ denote the point on py with parameter value t and let $s(t)$ denote its nearest point on S. Let the segment $ps(t)$ make an acute angle α with the tangent plane at p. See Figure 4. The rate of change of α with t is $\frac{d\alpha}{dt}$. In the limit this is same as $\frac{d}{dt}n(t)$ where $n(t)$ denotes the unit normal to S at $s(t)$. Thus, the total change in α over py is bounded by the total change in normal directions between p and y. It is proved in Lemma 2 of [1] that this change is at most $\frac{\delta}{1-3\delta}$ with $\delta < 1/3$, where $|py| \leq \delta \min\{f(p), f(y)\}$. Since $y \in V_p$, we have $|py| \leq \epsilon f(y)$ by sampling requirement. By Lipschitz condition $f(y) \leq f(p) + |py|$ giving $f(y) \leq \frac{f(p)}{1-\epsilon}$. Therefore $|py| \leq \epsilon f(y) \leq \frac{\epsilon}{1-\epsilon} f(p)$. We can safely take $\delta = \frac{\epsilon}{1-\epsilon}$ assuming $\epsilon < 0.25$. This gives $\frac{\delta}{1-3\delta} = \frac{\epsilon}{1-4\epsilon}$. ◻

Theorem 5 *For $\epsilon \leq 0.1$, all restricted Delaunay triangles are in T.*

Proof. Let e be the dual edge of a restricted Delaunay triangle. Consider the point $y = e \cap S$. We have $y \in V_{p,S}$ for some $p \in P$. We show that $\vec{y}$ makes an angle in the range I with the pole direction $\vec{v}$. The result follows from our choice of E. The acute angle between n_p and $\vec{y}$ is larger than $\pi/2 - \frac{\epsilon}{1-4\epsilon}$ by Lemma 4. Therefore $\angle\vec{y}\vec{v} \in [\pi/2 - \frac{\epsilon}{1-4\epsilon} - \alpha, \pi/2 + \frac{\epsilon}{1-4\epsilon} + \alpha]$, where α is the acute angle between $\vec{v}$ and n_p. Plugging in the upper bound of α from Lemma 3 we obtain that $\angle\vec{y}\vec{v} \in I$ when $\epsilon \leq 0.1$. ◻

3.2 Condition B

Interpreting Lemma 2 differently and plugging in a value of $\delta = \frac{2\epsilon}{1-\epsilon}$ we obtain the following.

Corollary 6 *Let x be any point in V_p so that the acute angle between $\vec{x}$ and n_p is at least $\pi/2 - 6\sin^{-1}\frac{\epsilon}{1-\epsilon}$. Then $|px| < \frac{2\epsilon}{1-\epsilon} f(p)$ for $\epsilon < 0.1$.*

Proof. If the acute angle between $\vec{x}$ and n_p is at least $\alpha = \sin^{-1}\frac{\epsilon}{\delta(1-\epsilon)} + \sin^{-1}\frac{\epsilon}{1-\epsilon}$, then $|px| < \delta f(p)$ according to Lemma 2. With $\delta = \frac{2\epsilon}{1-\epsilon}$ we have

$$\begin{aligned} \alpha &= \pi/6 + \sin^{-1}\frac{\epsilon}{1-\epsilon} \\ &\geq \pi/2 - 6\sin^{-1}\frac{\epsilon}{1-\epsilon} \end{aligned}$$

provided $7\sin^{-1}\frac{\epsilon}{1-\epsilon} \leq \pi/3$, which holds for $\epsilon \leq 0.1$. ◻

Theorem 7 *Let r denote the radius of any triangle $t \in T$. Then, for each vertex p of t, $r \leq \frac{2\epsilon}{1-\epsilon} f(p)$ for $\epsilon \leq 0.1$.*

Proof. Let e be the dual edge of t. By our choice of e, there is a point $x \in e$ so that $\vec{x}$ makes an angle in the range $I = [\pi/2 - 4\sin^{-1}\frac{\epsilon}{1-\epsilon}, \pi/2 + 4\sin^{-1}\frac{\epsilon}{1-\epsilon}]$ with $\vec{v}$. Taking into account the angle between $\vec{v}$ and n_p we conclude that this ray makes an acute angle more than $\pi/2 - 6\sin^{-1}\frac{\epsilon}{1-\epsilon}$ with n_p. From Corollary 6, $|px| < \frac{2\epsilon}{1-\epsilon} f(p)$. The radius of the circumcircle of t cannot be more than this. ◻

3.3 Condition C

Theorem 8 *Any triangle $t \in T$ has a normal that makes an acute angle no more than $\alpha + \sin^{-1}(\frac{2}{\sqrt{3}}\sin 2\alpha)$ with n_p where p is the vertex of t with the largest face angle and $\alpha \leq sin^{-1}\frac{2\epsilon}{1-\epsilon}$.*

Proof. Consider the medial balls M_1 and M_2 touching S at p with the centers on the medial axis. Let D be the ball with t lying on a diametric plane; refer to Figure 5. The radius r of D is equal to the radius of the circumcircle of t. Denote the circles of intersection of D with M_1 and M_2 as C_1 and C_2 respectively. The normal to S at p passes through m, the center of M_1. This normal makes an angle less than α with the normals to the planes of C_1 and C_2, where

$$\begin{aligned} \alpha &\leq \sin^{-1} r/|pm| \\ &\leq \sin^{-1}\frac{2\epsilon}{1-\epsilon} \end{aligned}$$

since $|pm| \geq f(p)$ by definition and $r \leq \frac{2\epsilon}{1-\epsilon} f(p)$ by Theorem 7. This angle bound also applies to the plane of C_2, which implies that the planes of C_1 and C_2 make a wedge, say W, with an acute dihedral angle no more than 2α.

The vertices q, s of t cannot lie inside M_1 or M_2. This implies that t lies completely in the wedge W. Consider a cone at p inside the wedge W formed by the three planes; π_t, the plane of t, π_1, the plane of C_1 and π_2, the plane of C_2. A unit sphere centered around p intersects the cone in a spherical triangle uvw, where u, v and w are the points of intersections of the lines $\pi_1 \cap \pi_2$, $\pi_t \cap \pi_1$ and $\pi_t \cap \pi_2$ respectively with the unit sphere. See the picture on right in Figure 5. Without the loss of generality, assume that the angle $\angle uvw \leq \angle uwv$. We have the following facts. The arc length of wv, denoted $|wv|$, is at least $\pi/3$ since p subtends the largest angle in t and t lies completely in the wedge W. The spherical angle $\angle vuw$ is less than or equal to 2α. We are interested in the spherical angle $\beta = \angle uvw$ which is also the acute dihedral angle between the planes of t and C_1. By standard *sine laws* in spherical geometry, we have $\sin\beta = \sin|uw|\frac{\sin\angle vuw}{\sin|wv|} \leq \sin|uw|\frac{\sin 2\alpha}{\sin|wv|}$. If $\pi/3 \leq |wv| \leq 2\pi/3$, we have $\beta \leq \sin^{-1}\frac{2}{\sqrt{3}}\sin 2\alpha$. For the range $2\pi/3 < |wv| < \pi$, we use the fact that $|uw| + |wv| \leq \pi$ since $\angle vuw \leq 2\alpha < \pi/2$ for sufficiently small ϵ. So, in this case $\frac{\sin|uw|}{\sin|wv|} < 1$. Thus, $\beta \leq \sin^{-1}\frac{2}{\sqrt{3}}\sin 2\alpha$.

The normals to t and S at p make an acute angle at most $\alpha + \beta$ proving the lemma. ◻

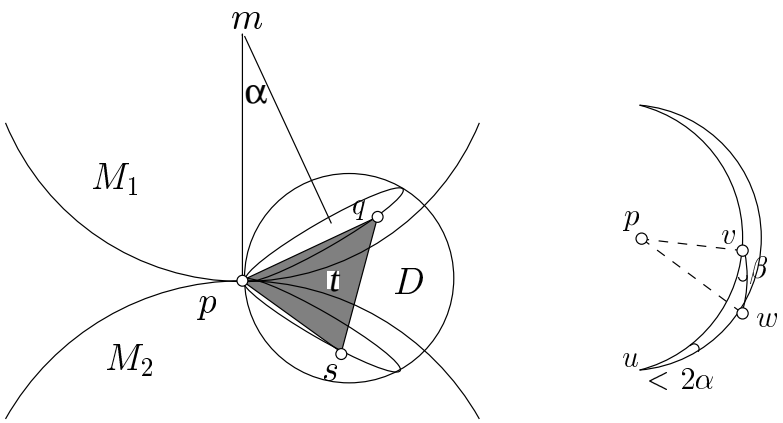

Figure 5: Normal to a small triangle and the normal to S at the vertex with the largest face angle.

4 Implementation

We have implemented the algorithm. Some outputs are shown in the next page. The algorithm was quite easy to implement with the well known *qhull* code for Voronoi diagram computation. In the figure we show the output of COMPUTET. The surface is computed correctly almost everywhere except at the boundaries and near sharp features. This is expected since the algorithm is not geared to handle boundaries or sharp features. However, the mechanical part has been computed reasonably well although it has sharp features. The program runs fast too. The program ran on a SUN machine with the 300Mhz processor and 256 MB memory for less than a couple of minutes on all the examples. The number of output triangles and the running times are shown under each example. For example, the foot took 153 seconds. It is reported in [3] that the same data took 15 minutes on a SGI Onyx machine with 512 MB memory. We plan to compare the performance of our algorithm with those of [1, 4] on similar platforms. Effort is being made towards that goal.

5 Conclusions

We simplified the crust algorithms of [1, 4]. Similar to the original algorithm in [1], our algorithm computes an approximation of a surface from its sample points with the guarantee of homeomorphism and geometric proximity. The merits of our algorithm over the other two competitive algorithms are: (i) it requires only one Voronoi diagram computation as opposed to two such computations in [1]; (ii) it collects a possible set of triangles from the Delaunay triangulations by checking a single condition as opposed to two such conditions in [4]; (iii) the proofs are simpler than [1, 4]; (iv) the constants derived in the proofs are smaller than [1, 4]. Our theory is supported by the output of our program on several data.

Acknowledgement

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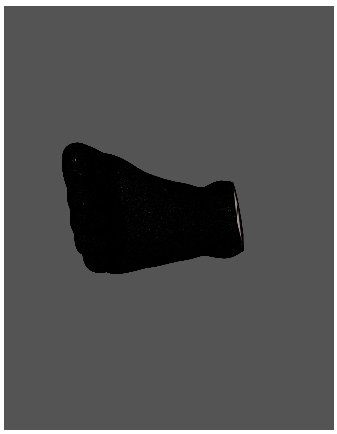
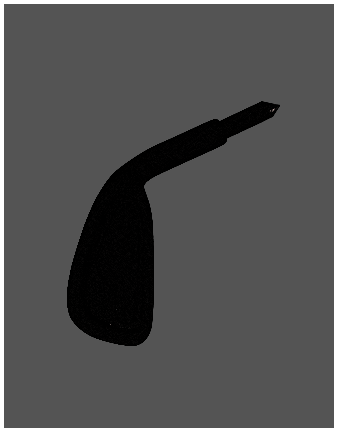
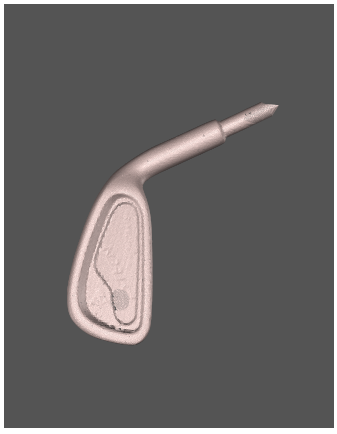


Foot, 50,341 triangles
Computing time 153 seconds

Smooth rendering



Club, 33,692 triangles
Computing time 122 seconds



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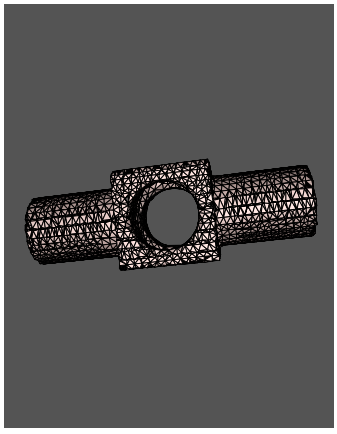
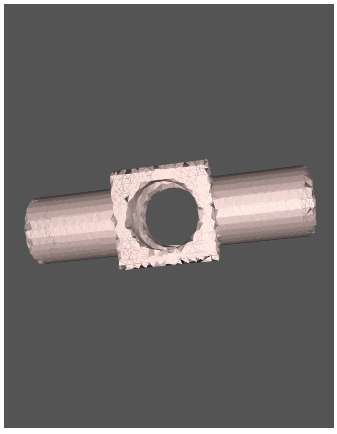


Mechanical part, 7,916 triangles
Computing time 44 seconds



Smooth rendering

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