30 Curve and Surface Reconstruction

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INTRODUCTION

The problem of reconstructing a shape from its sample appears in many scientific and engineering applications. Because of the variety in shapes and applications, many algorithms have been proposed over the last two decades, some of which exploit application-specific information and some of which are more general. We will concentrate on techniques that apply to the general setting and have proved to provide some guarantees on the quality of reconstruction.

GLOSSARY

Simplex: A k-simplex in $\mathbb{R}^d$, $0 \leq k \leq d$, is the convex hull of $k+1$ affinely independent points in $\mathbb{R}^d$ where $0 \leq k \leq d$. The 0-, 1-, 2-, and 3-simplices are also called *vertices*, *edges*, *triangles*, and *tetrahedra* respectively.

Simplicial complex: A simplicial complex $\mathcal{K}$ is a collection of simplices with the conditions that, (i) if $\sigma_1, \sigma_2 \in \mathcal{K}$ intersect, then $\sigma_1 \cap \sigma_2 \in \mathcal{K}$, and (ii) all simplices spanned by the vertices of a simplex in $\mathcal{K}$ are also in $\mathcal{K}$. The underlying space of $\mathcal{K}$ is the set of all points in its simplices. (Cf. Chapter 31.)

k-manifold: A k-manifold is a topological space where each point has a neighborhood homeomorphic to $\mathbb{R}^k$ or the halfspace $\mathbb{H}^k$. The points with $\mathbb{H}^k$ neighborhood constitute the boundary of the manifold.

Voronoi diagram: Given a point set $P \in \mathbb{R}^d$, the Voronoi diagram V_P of P is a collection of Voronoi cells V_p for each point $p \in P$, where

$$V_p = \{x \in \mathbb{R}^d \mid \|x - p\| \leq \|x - q\| \, \forall q \in P\} .$$

Delaunay triangulation: The Delaunay triangulation of a point set $P \in \mathbb{R}^d$ is a simplicial complex D_P so that a simplex with vertices $\{p_0, .., p_k\}$ is in D_P if and only if $\bigcap_{i=0,k} V_{p_i} \neq \emptyset$. (Cf. Chapter 22.)

Shape: A shape Σ is a subset of an Euclidean space.

Sample: A sample P of a shape Σ is a finite set of points from Σ.

Medial axis: The medial axis of a shape $\Sigma \in \mathbb{R}^d$ is the closure of the set of points in $\mathbb{R}^d$ that have more than one closest point in Σ. See Figure 30.1.1(a) for an illustration.

Local feature size: The local feature size for a shape $\Sigma \subseteq \mathbb{R}^d$ is a continuous function $f : \Sigma \to \mathbb{R}$ where $f(x)$ is the distance of $x \in \Sigma$ to the medial axis of Σ. See Figure 30.1.1(a).

Uniform sample: A sample P of a shape Σ is δ-uniform if for each $x \in \Sigma$ there is a sample point $p \in P$ so that $\|p - x\| \leq \delta f_{min}$ where $f_{min} = \min\{f(x), x \in \Sigma\}$ and $\delta > 0$ is a constant.

ϵ-sample: A sample P of a shape Σ is an ϵ-sample if for each $x \in \Sigma$ there is a sample point $p \in P$ so that $\|p - x\| \leq \epsilon f(x)$.

30.1 Curve Reconstruction

In its simplest form the reconstruction problem appeared in applications such as pattern recognition (Chapter 51), computer vision, and cluster analysis, where a curve in two dimensions is to be approximated from a set of sample points. In the 1980's several geometric graphs connecting a set of points in plane were discovered which reveal a pattern among the points. The influence graph of Toussaint [AH85], the β-skeleton of Kirkpatrick and Radke [KR85], the α-shapes of Edelsbrunner, Kirkpatrick, Seidel [EKS83] are such graphs. Recently, several algorithms have been proposed that reconstruct a curve from its sample with guarantees under some sampling assumption.

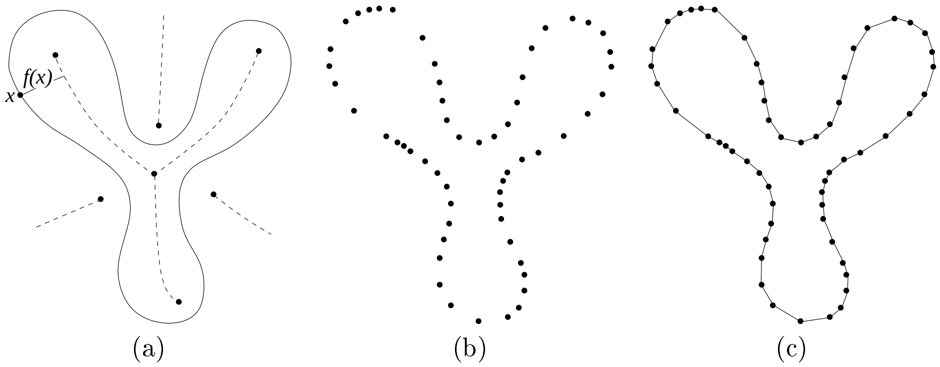

FIGURE 30.1.1
A smooth curve (solid), its medial axis (dashed) (a), sample (b), reconstruction (c).

GLOSSARY

Curve: A curve C in plane is a trace of a function $p : \mathbb{R} \to \mathbb{R}^2$ where $p(t) = (x(t), y(t))$ for $t \in [0,1]$ and $p[t] \neq p[t']$ for any $t \neq t'$ except possibly $t, t' \in \{0,1\}$. It is *smooth* if p is differentiable and the derivative $\frac{d}{dt}p(t) = (\frac{dx(t)}{dt}, \frac{dy(t)}{dt})$ does not vanish.

Boundary: A curve C is said to have no boundary if $p[0] = p[1]$; otherwise, it is a curve with boundary.

Reconstruction: The reconstruction of C from its sample P is a geometric graph $G = (P, E)$ where an edge pq belongs to E if and only if p and q are adjacent sample points on C. See Figure 30.1.1.

Semi-regular curve: One for which the left tangent and right tangent exist at each point of the curve, though they may be different.

Uniform Sample

α-shapes: Edelsbrunner, Kirkpatrick, and Seidel [EKS83] defined the α-shape of a point set $P \subseteq \mathbb{R}^2$ as the underlying space of a simplicial complex called the α-complex. The α-complex of P is defined by all simplices with vertices in P that have an empty circumscribing disk of radius α. Bernardini and Bajaj [BB97] show that the α-shapes reconstruct curves from δ-uniform samples if δ is sufficiently small and α is chosen appropriately.

r-regular shapes: Attali considered r-regular shapes that are constructed using certain morphological operations with r as a parameter [Att97]. It turns out that these shapes are characterized by requiring that any circle passing through the points on the boundary has radius greater than r. A sample P from the boundary curve C of such a shape is called γ-sample if each point $x \in C$ has a sample point within γr distance. Let η_{pq} be the sum of the angles opposite to pq in the two incident Delaunay triangles at a Delaunay edge $pq \in D_P$. The main result in [Att97] is that if $\gamma < \sin\frac{\pi}{8}$, Delaunay edges with $\eta_{pq} < \pi$ reconstruct C.

EMST: Figueiredo and Gomes [FG95] show that the Euclidean minimum spanning tree (EMST) reconstructs curves with boundaries when the sample is sufficiently dense. The sampling density condition that is used to prove this result is equivalent to that of δ-uniform sampling for an appropriate $\delta > 0$. Of course, EMST cannot reconstruct curves without boundaries and/or multiple components.

Nonuniform Sample

Crust: Amenta, Bern, and Eppstein [ABE98] proposed the first algorithm to reconstruct a curve from a non-uniform sample with guarantee. The algorithm computes the *crust* of P in two phases. The first phase computes the Voronoi diagram of the sample points in P. Let V be the set of Voronoi vertices in this diagram. The second phase computes the Delaunay triangulation of the larger set $P \cup V$. The Delaunay edges that connect only sample points in this triangulation constitute the crust; see Figure 30.1.2.

The theoretical guarantee of the crust algorithm is based on the notion of dense sampling that respects features of the sampled curve. The important concepts of local feature size and ϵ-sampling were introduced by Amenta, Bern and Eppstein [ABE98]. They prove that if P is an ϵ-sample of a curve C without boundary for $\epsilon \leq 0.252$, the crust reconstructs C. The two Voronoi diagram computations of the crust are reduced to one by Gold and Snoeyink [GS01].

Nearest neighbor: After the introduction of the crust, Dey and Kumar [DK99] proposed a curve reconstruction algorithm based on nearest neighbors. They showed that all nearest neighbor edges that connect a point to its Euclidean nearest neighbor must be in the reconstruction if the input is $\frac{1}{3}$-sample. However, not all edges of the reconstruction are necessarily nearest neighbor edges. The remaining edges are characterized as follows. Let p be a sample point with only one nearest neighbor edge pq incident to it. Consider the halfplane with pq being an outward normal to its bounding line through p, and let r be the nearest to p among all sample points

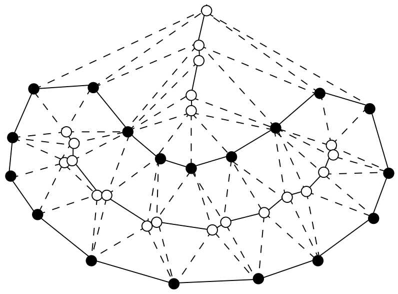

FIGURE 30.1.2
Crust edges (solid) among the Delaunay triangulation of a sample and their Voronoi vertices.

lying in this halfplane. Call pr the half-neighbor edge of p. Dey and Kumar show that all half-neighbor edges must also be in the reconstruction for a $\frac{1}{3}$-sample.

The algorithm first computes all nearest neighbor edges and then computes the half-neighbor edges to complete the reconstruction. Since all edges in the reconstruction must be a subset of Delaunay edges if the sample is sufficiently dense, all nearest neighbor and half-neighbor edges can be computed from the Delaunay triangulation. Thus, as crust this algorithm runs in $O(n \log n)$ time for a sample of n points.

Nonsmoothness, Boundaries

The crust and nearest neighbor algorithms assume that the sampled curve is smooth and has no boundary. Nonsmoothness and boundaries make reconstruction harder.

Traveling Salesman Path: Giesen [Gie00] considered a fairly large class of nonsmooth curves and showed that Traveling Salesman Path (or Tour) reconstructs them from sufficiently dense samples. A semi-regular curve C is *benign* if the angle between the two tangents at each point is less than π. Giesen proved that, for a benign curve C, there exists a $\delta > 0$ so that if each $x \in C$ has a sample point p with $\|p - x\| \leq \delta$, then C is reconstructed by the Traveling Salesman Path (or Tour) in case C has boundary (or no boundary).

The uniform sampling condition for the Traveling Salesman approach was later removed by Althaus and Mehlhorn [AM02], who also gave a polynomial-time algorithm to compute the Traveling Salesman Path (or Tour) in this special case of curve reconstruction. Obviously, the Traveling Salesman approach cannot handle curves with multiple components. Also, the sample points representing the boundary need to be known a priori to choose between path or tour.

Conservative Crust: In order to allow boundaries in curve reconstruction, it is essential that the sample points representing boundaries are detected. Dey, Mehlhorn and Ramos presented such an algorithm, called the *conservative crust* [DMR00].

Any algorithm for handling curves with boundaries faces a dilemma when an input point set samples a curve without boundary densely and simultaneously samples another curve with boundary densely. This dilemma is resolved in conservative crust by a justification on the output. For any input point set P, the graph output by the algorithm is guaranteed to be the reconstruction of a smooth curve C possibly with boundary for which the input point set is a dense sample. The main idea of the algorithm is that an edge pq is chosen in the output only if there is a large enough ball centering the midpoint of pq which is empty of all Voronoi vertices in the Voronoi diagram of P. The rationale behind this choice is that these edges are small enough with respect to local feature size of C since the Voronoi vertices approximate its medial axis.

With certain sampling condition tailored to handle nonsmooth curves, Funke and Ramos used conservative crust to reconstruct nonsmooth curves that may have boundaries [FR01].

SUMMARIZED RESULTS

The strengths and deficiencies of the discussed algorithms are summarized in Table 30.1.1.

TABLE 30.1.1 Curve reconstruction algorithms.

ALGORITHM	SAMPLE	SMOOTHNESS	BOUNDARY	COMPONENTS
α-shape	uniform	required	none	multiple
r-regular shape	uniform	required	none	multiple
EMST	uniform	required	exactly two	single
Crust	non-uniform	required	none	multiple
Nearest neighbor	non-uniform	required	none	multiple
Traveling Salesman	non-uniform	not required	must be known	single
Conservative crust	non-uniform	required	any number	multiple

OPEN PROBLEM

All algorithms described above assume that the sampled curve does not cross itself. It is open to devise an algorithm that can reconstruct such curves under some reasonable sampling condition.

30.2 Surface Reconstruction

A number of surface reconstruction algorithms have been designed in different application fields in recent years. The problem appeared in medical imaging where a set of cross sections obtained via CAT scan or MRI needs to be joined with a surface. The points on the boundary of the cross sections are already joined by a polygonal curve and the output surface needs to join these curves in consecutive cross sections. A dynamic programming based solution for two such consecutive curves was first proposed by Fuchs, Kedem and Uselton [FKU77]. A negative result by Gitlin, O'Rourke and Subramanian [GOS96] shows that, in general, two polygonal curves cannot be joined by a nonself-intersecting surface with only those vertices; even deciding its possibility is NP-hard. Several solutions with the addition of Steiner points have been proposed to overcome the problem, see [MSS92, BG93]. The most general version of the surface reconstruction problem does not assume any information about the input points other than their 3D coordinates, and requires a piecewise linear approximation of the surface from which the input point sample is derived; see Figure 30.2.1. In the context of computer graphics and vision, this problem has been investigated intensely in the past decade with emphasis on the practical effectiveness of the algorithms [BMR$^+$99, Boi84, CL96, GKS00, HDD$^+$92]. Lately, several algorithms have been designed mainly based on Voronoi/Delaunay diagrams that have theoretical guarantees. We focus mainly on them.

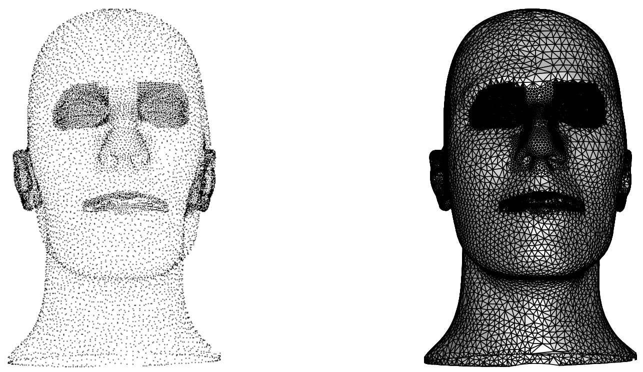

FIGURE 30.2.1
A point sample and the reconstructed surface.

GLOSSARY

Surface: A surface $S \subset \mathbb{R}^3$ is a 2-manifold embedded in $\mathbb{R}^3$. Thus each point $p \in S$ has a neighborhood homeomorphic to $\mathbb{R}^2$ or halfplane $\mathbb{H}^2$. The points with neighborhoods homeomorphic to $\mathbb{H}^2$ constitute the boundary of S.

Smooth Surface: A surface $S \subset \mathbb{R}^3$ is smooth if for each point $p \in S$ there is a neighborhood $W \subseteq \mathbb{R}^3$ and a map $\pi : U \to W \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $W \cap S$ so that

(i)π is differentiable,

(ii)π is a homeomorphism,

(iii)for each $q \in U$ the differential $d\pi_q$ is one-to-one.

Restricted Voronoi: Given a subspace $\mathbb{N} \subseteq \mathbb{R}^3$ and a point set $P \subseteq \mathbb{R}^3$, the restricted Voronoi diagram of V_P with respect to $\mathbb{N}$ is $V_{P,\mathbb{N}} = V_P \cap \mathbb{N}$.

Restricted Delaunay: The dual of $V_{P,\mathbb{N}}$ is called the restricted Delaunay triangulation $D_{P,\mathbb{N}}$ defined as

$$D_{P,\mathbb{N}} = \{\sigma \,|\, \sigma = (p_0, ..., p_k) \in D_P \text{ where } (\cap_{i=0,k} V_{p_i}) \bigcap \mathbb{N} \neq \emptyset\}.$$

Watertight surface: A 2-complex $\mathcal{K}$ embedded in $\mathbb{R}^3$ is called watertight if the underlying space of $\mathcal{K}$ is same as the boundary of the closure of a 3-manifold in $\mathbb{R}^3$.

Steiner points: The points used by an algorithm that are not part of the finite input point set are called Steiner points.

α-shapes

Generalization of α-shapes to 3D by Edelsbrunner and Mücke [EM94] can be used for surface reconstruction in case the sample is more or less uniform. An alternate definition of α-shapes in terms of the restricted Delaunay triangulation is more appropriate for surface reconstruction. Let $\mathbb{N}$ denote the space of all points covered by open balls of radius α around each sample point $p \in P$. The α-shape for P is the underlying space of the restricted Delaunay triangulation $D_{P,\mathbb{N}}$; see Figure 30.3.1 below for an illustration in 2D. It is shown that the α-shape is always homotopic to $\mathbb{N}$, which in turn is homotopic to S if α is chosen appropriately for a sufficiently dense P [EM94]. Therefore, by transitivity of homotopy maps, the α-shape is homotopic to S if α is appropriate and the sample P is sufficiently dense.

The major drawback of α-shapes is that it requires a nearly uniform sample for reconstruction, and the value of α must be chosen appropriately. In a work under proprietary rights Edelsbrunner designed the WRAP algorithm based on Morse theory that overcomes the shortcoming of α-shapes [Ede03].

Crust

The crust algorithm for curve reconstruction was generalized for surface reconstruction by Amenta and Bern [AB99]. In case of curves in 2D, Voronoi vertices for a dense sample lie close to the medial axis. That is why a second Voronoi diagram with the input sample points together with the Voronoi vertices is used to separate the Delaunay edges that reconstruct the curve. Unfortunately, Voronoi vertices in 3D can lie arbitrarily close to the sampled surface. One can place four arbitrarily close points on a smooth surface which lie near the diametric plane of the sphere defined by them. This sphere can be made empty of any other input point and thus its center as a Voronoi vertex lies close to the surface. With this important observation Amenta and Bern forsake the idea of putting all Voronoi vertices in the second phase of crust and instead identify a subset of Voronoi vertices called ***poles*** that lie far away from the surface, and in fact close to the medial axis.

Let P be an ϵ-sample of a compact smooth surface S without boundary. Let V_p be a Voronoi cell in the Voronoi diagram V_P. The farthest Voronoi vertex of V_p from p is called the positive pole of p. Call the vector from p to the positive pole the ***pole vector*** for p; this vector approximates the surface normal $\mathbf{n}_p$ at p. The Voronoi vertex of V_p that lies farthest from p in the opposite direction of the pole vector is called its negative pole. The opposite direction is specified by the condition that the vector from p to the negative pole must make an angle more than $\frac{\pi}{2}$ with the pole vector. Figure 30.2.2(a) illustrates these definitions. If V_p is unbounded, the positive pole is taken at infinity and the direction of the pole vector is taken as the average of all directions of the unbounded Voronoi edges in V_p.

The crust algorithm in 3D proceeds as follows. First, it computes the Voronoi diagram V_P and then identifies the set of poles, say L. The Delaunay triangulation of the point set $P \cup L$ is computed and the set of Delaunay triangles, T, is filtered that have all three vertices only from P. This set of triangles almost approximates S but may not form a surface. Nevertheless, the set T includes all restricted Delaunay triangles in $D_{P,S}$. According to a result by Edelsbrunner and Shah [ES97], $D_{P,S}$ is homeomorphic to S if each Voronoi cell satisfies a topological condition called the "closed ball property." Amenta and Bern show that if P is an ϵ-sample for $\epsilon \leq 0.06$, each Voronoi cell in V_P satisfies this property. This means that, if the triangles in $D_{P,S}$ can be extracted from T, we will have a surface homeomorphic to S. Unfortunately, it is impossible to detect the restricted Delaunay triangles of $D_{P,S}$ since S is unknown. However, the fact that T contains them is used in extracting a manifold out of T after a normal filtering step. This piecewise linear manifold surface is output as crust.

The crust guarantees that the output surface lies very close to S. In particular, each point p in the output has a point x in S so that $\|p - x\| \leq O(\epsilon)f(x)$. Also, each point x in S has a point p in the output so that the same bound holds.

Cocone

The cocone algorithm was developed from the crust algorithm by Amenta, Choi, Dey and Leekha [ACDL02]. It simplified the reconstruction algorithm and its proof of correctness.

A ***cocone*** C_p for a sample point p is defined as the complement of the double cone with p as apex and the pole vector as axis and an opening angle of $\frac{3\pi}{4}$; see Figure 30.2.2(b). Because the pole vector at p approximates the surface normal $\mathbf{n}_p$, the cocone C_p (clipped within V_p) approximates a thin neighborhood around the tangent plane at p. For each point p, the algorithm then determines all Voronoi edges in V_p that are intersected by the cocone C_p. The dual Delaunay triangles of these Voronoi edges constitute the set of candidate triangles T.

It is shown that the circumscribing circles of all candidate triangles are small [ACDL02]. Specifically, if $pqr \in T$ has circumradius r, then

$$(i)\ r = O(\epsilon)f(x) \text{ where } f(x) = \min\{f(p), f(q), f(r)\}.$$

It turns out that any triangle with such small circumradius must lie flat to the surface, i.e., if $\mathbf{n}_{pqr}$ is the normal to a candidate triangle pqr, then

$$(ii)\ \angle(\mathbf{n}_{pqr}, \mathbf{n}_x) = O(\epsilon) \text{ up to orientation where } x \in \{p, q, r\}.$$

Also, it is proved that

$$(iii)\ T \text{ includes all restricted Delaunay triangles in } D_{P,S}.$$

These three properties of the candidate triangles ensure that a manifold extraction step, as in crust algorithm, extracts a piecewise-linear surface N which is homeomorphic to the original surface S.

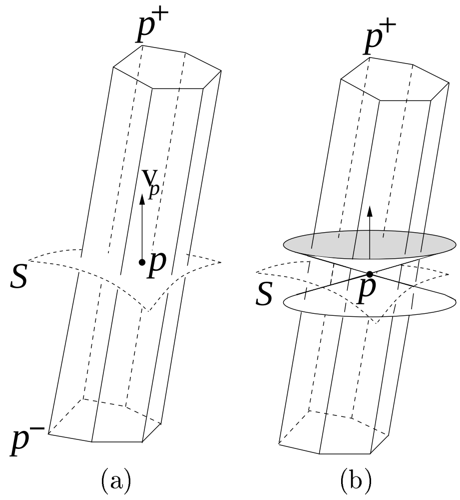

FIGURE 30.2.2
A long thin Voronoi cell V_p, the positive pole p^+, the pole vector $\mathbf{v}_p$ and the negative pole p^- (a), the cocone (b).

Cocone uses a single Voronoi diagram as opposed to two in the crust algorithm and also eliminates the normal filtering step. It guarantees that the output surface N is topologically equivalent to the sampled surface S for $\epsilon \leq 0.06$ and each point on N has a point x in S within $O(\epsilon)f(x)$ distance. Because of the Voronoi diagram computation, the cocone runs in $O(n^2)$ time and space. Funke and Ramos [FR02] improved its complexity to $O(n \log n)$ though the resulting algorithm seems impractical.

Natural Neighbor

Boissonnat and Cazals [BC00] revisited the approach of Hoppe et al. [HDD$^+$92] by approximating the sampled surface as the zero set of a signed distance function. They used natural neighbors and ϵ-sampling to provide output guarantees.

Given an input point set $P \subset \mathbb{R}^3$, the ***natural neighbors*** $N_{x,P}$ of a point $x \in \mathbb{R}^3$ are the Delaunay neighbors of x in $D_{P \cup x}$. Letting $V(x)$ denote the Voronoi cell of x in $V_{P \cup x}$, this means

$$N_{x,P} = \{p \in P \mid V(x) \cap V_p \neq \emptyset\}.$$

Let $A(x,p)$ denote the volume stolen by x from V_p, i.e.,

$$A(x,p) = V(x) \cap V_p.$$

The natural coordinate associated with a point p is a continuous function $\lambda_p : \mathbb{R}^3 \to \mathbb{R}$ where

$$\lambda_p(x) = \frac{A(x,p)}{\Sigma_{q \in P} A(x,q)}.$$

Some of the interesting properties of λ_p are that it is continuously differentiable except at p, and any point $x \in \mathbb{R}^3$ is a convex combination of its natural neighbors:

$\Sigma_{p \in N_{x,P}} \lambda_p(x) p = x$. Boissonnat and Cazals assume that each point p is equipped with a unit normal $\mathbf{n}_p$ which can either be computed via pole vectors, or is part of the input. A distance function $h_p : \mathbb{R}^3 \to \mathbb{R}$ for each point p is defined as $h_p(x) = (p - x) \cdot \mathbf{n}_p$. A global distance function $h : \mathbb{R}^3 \to \mathbb{R}$ is defined by interpolating these local distance functions with natural coordinates. Specifically,

$$h(x) = \Sigma_{p \in P} \lambda_p^{1+\delta}(x) h_p(x).$$

The δ term in the exponent is added to make h continuously differentiable. By definition, $h(x)$ locally approximates the signed distance from the tangent plane at each point $p \in P$ and, in particular, $h(p) = 0$.

Since h is continuously differentiable, $\hat{S} = h^{-1}(0)$ is a smooth surface unless 0 is a critical value. A discrete approximation of $\hat{S}$ can be computed from the Delaunay triangulation of P as follows. All Voronoi edges that intersect $\hat{S}$ are computed via the sign of h at their two endpoints. The dual Delaunay triangles of these Voronoi edges constitute a piecewise linear approximation of $\hat{S}$. If the input sample P is an ϵ-sample for sufficiently small ϵ, then it can be shown that the output surface is geometrically close and is also topologically equivalent to the sampled surface.

Undersampling

The assumption of ϵ-sampling for sufficiently small $\epsilon > 0$ often does not hold in practice. This undersampling may be caused by inadequate attention during sampling process, or machine error, or nonsmoothness. Even boundaries in a surface may be viewed as the demarcation where appropriate sampling stops and the undersampling begins. Dey and Giesen [DG01] took this unified view to detect boundaries that identify the regions of undersampling.

Given a sample P of a surface S, the subset $S' \subseteq S$ is called ***well-sampled*** if each point x in S' has a sample point within $\epsilon f(x)$ distance. If P undersamples S, the well-sampled surface S' has boundaries. A point $p \in P$ is a ***boundary sample*** if V_p intersects the boundary of S', otherwise p is *interior*. The algorithm of Dey and Giesen [DG01] works in two phases to detect all boundary samples. In the first phase, it selects a set $R \subseteq P$ based on two conditions. The first condition requires the Voronoi cell of a point in R be long and thin and the second requires its pole vector agree with those of all its neighbors on the surface (determined by cocones). These two conditions ensure that R consists of interior points only. In a second phase, the set R is expanded to include more points by relaxing the second condition. It is proved that under some mild assumptions on sampling, this algorithm determines all interior points and the remaining points are correctly detected as boundary.

Once the boundary samples are detected, the cocone algorithm is employed to filter the candidate triangles. Boundary samples are not allowed to choose any triangle. This produces the boundaries at the undersampled regions.

Watertight Surfaces

Most of the surface reconstruction algorithms face a difficulty while dealing with undersampled surfaces and noise. While the algorithm of [DG01] can detect undersampling, it leaves holes in the surface near the vicinity of undersampling. Although this may be desirable for reconstructing surfaces with boundaries, many applications such as CAD designs require that the output surface be ***watertight***, i.e., a surface that bounds a solid.

The natural neighbor algorithm of [BC00] can be adapted to guarantee a watertight surface. Recall that this algorithm approximates a surface $\hat{S}$ implicitly defined by the zero set of a smooth map $h : \mathbb{R}^3 \to \mathbb{R}$. This surface is a smooth 2-manifold without boundary in $\mathbb{R}^3$. However, if the input sample P is not dense for this surface, the reconstructed output may not be watertight. Boissonnat and Cazals suggest to sample more points on $\hat{S}$ to obtain a dense sample for $\hat{S}$ and then reconstruct it from the new sample.

Amenta, Choi and Kolluri [ACK01] use the crust approach to design the ***power crust*** algorithm to produce watertight surfaces. This algorithm first distinguishes the inner poles that lie inside the solid bounded by the sampled surface S from the outer poles that lies outside. A consistent orientation of the pole vectors is used to decide between inner and outer poles. To prevent outer poles at infinity, eight corners of a large box containing the sample are added. The union of Delaunay balls with centers at the inner poles approximate the solid bounded by S. The union of Delaunay balls centered at the outer poles do not approximate the entire exterior of S although one of its boundary component approximates S. The implication is that the cells in the power diagram of the poles with the radius of the Delaunay ball as weights can be partitioned into two sets, with the boundary between approximating S. The facets in the power diagram that separate cells generated by inner poles from the ones generated by outer poles form this boundary which is output by power crust.

Recently Dey and Goswami [DG02] announced a water-tight surface reconstructor called ***tight cocone***. This algorithm first computes the surface with cocone. Recall that cocone may leave some holes in the surface due to undersampling. A subsequent sculpting [Boi84] in the Delaunay triangulation of the input points recover triangles that fill the holes. Unlike power crust, tight cocone does not add Steiner points.

SUMMARIZED RESULTS

The properties of the above discussed surface reconstruction algorithms are summarized in Table 30.2.1.

OPEN PROBLEMS

All guarantees given by various surface reconstruction algorithms depend on the notion of dense sampling. Watertight surface algorithms can guarantee a surface without holes, but no theoretical guarantees exists under any type of undersampling.

1. Design an algorithm that reconstructs nonsmooth surfaces under reasonable sampling conditions.
2. Design a surface reconstruction algorithm that handles noise gracefully, and with some guarantees.

TABLE 30.2.1 Surface reconstruction algorithms.

ALGORITHM	SAMPLE	PROPERTIES	SOURCE
α-shape	uniform	α to be determined.	[EM94]
Crust	non-uniform	Theoretical guarantees from Voronoi structures, two Voronoi computations.	[AB99]
Cocone	non-uniform	Simplifies crust, single Voronoi computation with topological guarantee, detects undersampling	[ACDL02] [DG01]
Natural Neighbor	non-uniform	Theoretical guarantees using Voronoi diagram and implicit functions.	[BC00]
Power Crust	non-uniform	Watertight surface using power diagrams, introduces Steiner points.	[ACK01]
Tight Cocone	non-uniform	Watertight surface using Delaunay triangulation.	[DG02]

30.3 Shape reconstruction

All algorithms discussed above are designed for reconstructing a shape of specific dimension from the samples. Thus, the curve reconstruction algorithms cannot handle samples from surfaces and the surface reconstruction algorithms cannot handle samples from curves. Therefore, if a sample is derived from shapes of mixed dimensions, i.e., both curves and surfaces in $\mathbb{R}^3$, none of the curve and surface reconstruction algorithms is adequate. General shape reconstruction algorithms should be able to handle any shape embedded in Euclidean spaces. However, this goal may be too ambitious, as it is not clear what would be a reasonable definition of dense samples for general shapes that are nonsmooth or nonmanifold. The ϵ-sampling condition would require infinite sampling in these cases. We therefore distinguish two cases: (i) smooth manifold reconstruction for which a computable sampling criterion can be defined, (ii) shape reconstruction for which it is currently unclear how a computable sampling condition could be defined to guarantee reconstruction. This leads to a different definition for the general shape reconstruction problem in the glossary below.

GLOSSARY

Shape reconstruction: Given a set of points $P \subseteq \mathbb{R}^{\mathrm{d}}$, compute a shape that best approximates P.

Manifold shape: A manifold shape is a collection of smooth manifolds $\{M_1, M_2, ..., M_\ell\}$ embedded in an Euclidean space $\mathbb{R}^{\mathrm{d}}$.

Manifold reconstruction: Compute a piecewise-linear approximation to each M_i, given a sample P from a manifold shape $\{M_1, M_2, ..., M_\ell\}$.

MAIN RESULTS

Shape reconstruction: Not many algorithms are known to reconstruct shapes. The definition of α-shapes is general enough to be applicable to shape reconstruction. In Figure 30.3.1, the α-shape reconstructs a shape in $\mathbb{R}^2$ which is not a manifold. Similarly, it can reconstruct curves, surfaces and solids and their combinations in three dimensions. Melkemi [Mel97] proposed $\mathcal{A}$-shapes that can reconstruct shapes in $\mathbb{R}^2$. Its class of shapes includes α-shapes. Given a set of points P in $\mathbb{R}^2$, a member in this class of shapes is identified with another finite set $\mathcal{A} \subseteq \mathbb{R}^2$. The $\mathcal{A}$-shape of S is generated by edges that connect points $p, q \in P$ if there is a circle passing through p, q and a point in $\mathcal{A}$, and all other points in $P \cup \mathcal{A}$ lie outside the circle. The α-shape is a special case of $\mathcal{A}$-shapes where $\mathcal{A}$ is the set of all points on Voronoi edges that span empty circles with points in P. The crust is also a special case of $\mathcal{A}$-shape where $\mathcal{A}$ is the set of Voronoi vertices.

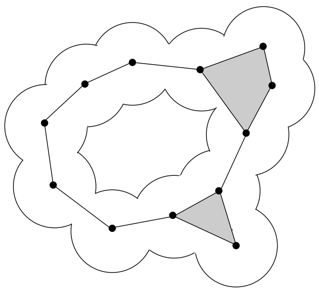

FIGURE 30.3.1
Alpha shape of a set of points in $\mathbb{R}^2$.

Manifold reconstruction: When the sample P derives from smooth manifolds embedded in some Euclidean space $\mathbb{R}^{\mathrm{d}}$, Dey et al. [DGGZ02] propose an algorithm *CoconeShape* for reconstruction. This algorithm first determines the dimension k of a sample point $p \in P$ if p is derived from a k-manifold. This dimension detection is accomplished by analyzing the structure of the Voronoi cell V_p. Subsequent to the dimension detection, a subset of k-dimensional Delaunay simplices incident to p are chosen in an output set T. This computation is performed with a generalized concept of cocones.

It is shown that the underlying space of T lies very close to the sampled manifold(s) although it may not be a triangulation of a manifold. A manifold extraction step as in the case of surfaces in $\mathbb{R}^3$ is necessary to clean T, but it is not clear how to do this effectively. In $\mathbb{R}^2$ and $\mathbb{R}^3$, the manifold extraction step can be performed and hence the manifold reconstruction problem is solved in $\mathbb{R}^2$ and $\mathbb{R}^3$.

OPEN PROBLEMS

1. Design an algorithm that outputs manifolds approximating sampled manifold shapes in $\mathbb{R}^d$, $d \geq 4$.
2. Reconstruct shapes with guarantees.

30.4 SOURCES AND RELATED MATERIAL

SURVEYS

[Ede98]: Shape reconstruction with Delaunay complex.

[OR00]: Computational geometry column 38 (Recent results on curve reconstruction).

[MSS92]: Surfaces from contours.

[MM98]: Interpolation and approximation of surfaces from three-dimensional scattered data points.

RELATED CHAPTERS

Chapter 22: Voronoi Diagrams and Delaunay triangulations
Chapter 24: Triangulations and mesh generation
Chapter 28: Geometric reconstruction problems
Chapter 31: Computational topology
Chapter 49: Computer graphics
Chapter 51: Pattern recognition
Chapter 54. Surface simplification and 3D geometry compression

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