2.2 Crust

We have already seen that all correct edges connecting consecutive sample points in an ε-sample are present in the Delaunay triangulation of the sample points if $\varepsilon < 1$. The main algorithmic challenge is to distinguish these edges from the rest of the Delaunay edges. The CRUST algorithm achieves this by observing some properties of the Voronoi vertices.

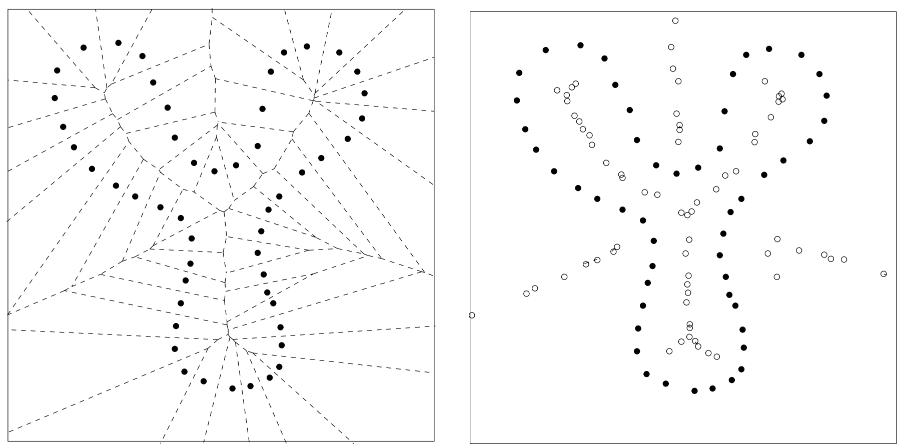

Figure 2.4: Voronoi vertices approximate the medial axis of a curve in the plane. The Voronoi vertices are shown with hollow circles in the right picture.

2.2.1 Algorithm

Consider Figure 2.4. The left picture shows the Voronoi diagram clipped within a box for a dense sample of a curve. The picture on the right shows the Voronoi vertices separately. A careful observation reveals that the Voronoi vertices lie near the medial axis of the curve (see exercise 3). The CRUST algorithm exploits this fact. All empty balls circumscribing incorrect edges in $\mathrm{Del}\, P$ cross the medial axis and hence contain Voronoi vertices inside. Therefore, they cannot appear in the Delaunay triangulation of P and V together where V is the set of Voronoi vertices in $\mathrm{Vor}\, P$. On the other hand, all correct edges still survive in $\mathrm{Del}\,(P \cup V)$. So, the algorithm first computes $\mathrm{Vor}\, P$ and then computes the Delaunay triangulation of $P \cup V$ where V is the set of Voronoi vertices of $\mathrm{Vor}\, P$. The Delaunay edges of $\mathrm{Del}\,(P \cup V)$ that connect two points in P are output. It is proved that an edge is output if and only if it is correct.

```
CRUST (P)
1  compute Vor P;
2  Let V be the Voronoi vertices of Vor P;
3  compute Del (P ∪ V);
4  E := φ;
5  for each edge pq ∈ Del (P ∪ V) do
6     if p ∈ P and q ∈ P
7        E := E ∪ pq
8     endif;
9  return E.
```

The Voronoi and the Delaunay diagrams of a set of n points in the plane can be computed in $O(n \log n)$ time and $O(n)$ space. The second Delaunay triangulation in step 3 deals with $O(n)$ points as the Voronoi diagram of n points can have at most $2n$ Voronoi vertices. Therefore, CRUST runs in $O(n \log n)$ time and takes $O(n)$ space.

2.2.2 Correctness

The correctness of CRUST is proved in two parts. First, it is shown that each correct edge is present in the output of CRUST (Correct Edge Lemma (2.6)). Then, it is shown that no incorrect edge is output (Incorrect Edge Lemma (2.7)).

Lemma 2.6 (Correct Edge.) *Each correct edge is output by* CRUST *when* $\varepsilon < \frac{1}{5}$.

PROOF. Let pq be a correct edge. Let z be the point where the perpendicular bisector of pq intersects the empty segment $\gamma(p,q)$. Consider the ball $B = B_{z,\|p-z\|}$. This ball is empty of any point from P when $\varepsilon < 1$ (Empty Segment Lemma (2.1) (i)). We show that this ball does not contain any Voronoi vertex of $\mathrm{Vor}\,P$ either.

Suppose that B contains a Voronoi vertex, say v, from V (Figure 2.5). Then, by simple circle geometry, the maximum distance of v from p is $2\|p-z\|$ which by Empty Segment Lemma (2.1) (iii) implies $\|p-v\| \le \frac{2\varepsilon}{1-\varepsilon} f(p)$. The Delaunay ball B' centering v contains three points from P on its boundary. This means $\mathrm{bd}\, B' \cap \Sigma$ is not a 0-sphere. So, B' contains a medial axis point by Feature Ball Lemma (1.1). As the Delaunay ball B' is empty, p cannot lie in $\mathrm{Int}\, B'$. So, the medial axis point in B' lies within $2\|p-v\|$ distance

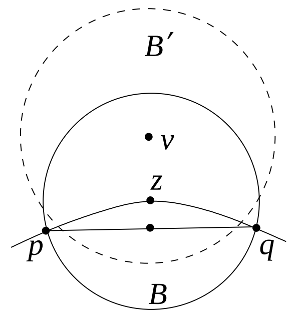

Figure 2.5: Illustration for Correct Edge Lemma

from p. Therefore, $2\|p - v\| \geq f(p)$. But, $\|p - v\| \leq \frac{2\varepsilon}{1-\varepsilon} f(p)$ enabling us to reach a contradiction when $\frac{2\varepsilon}{1-\varepsilon} < \frac{1}{2}$, i.e., when $\varepsilon < \frac{1}{5}$.

Therefore, for $\varepsilon < \frac{1}{5}$, there is a circumscribing ball of pq empty of any point from $P \cup V$. So, it appears in $\mathrm{Del}\,(P \cup V)$ and is output by CRUST as it connects two points from P. □

Lemma 2.7 (Incorrect Edge.) *No incorrect edge is output by* CRUST *when* $\varepsilon < 1/5$.

PROOF. We need to show that there is no ball, empty of both sample points and Voronoi vertices, circumscribing an incorrect edge between two sample points, say p and q. For the sake of contradiction, assume that D is such a ball.

Let v and v' be the two points where the perpendicular bisector of pq intersects the boundary of D, see Figure 2.6. Consider the two balls $B = B_{v,r}$ and $B' = B_{v',r'}$ that circumscribe pq.

We claim that both B and B' are empty of any sample points. Suppose on the contrary, any one of them, say B, contains a sample point. Then, one can push D continually towards B by moving its center on the perpendicular bisector of pq and keeping p, q on its boundary. During this motion, the deformed D would hit a sample point s for the first time before its center reaches v. At that moment p, q and s define a ball empty of any other sample points. The center of this ball is a Voronoi vertex in $\mathrm{Vor}\,P$ which resides inside D. This is a contradiction as D is empty of any Voronoi vertex from V.

The angle $\angle vpv'$ is $\pi/2$ as vv' is a diameter of D. The tangents to the boundary circles of B and B' at p are perpendicular to vp and $v'p$

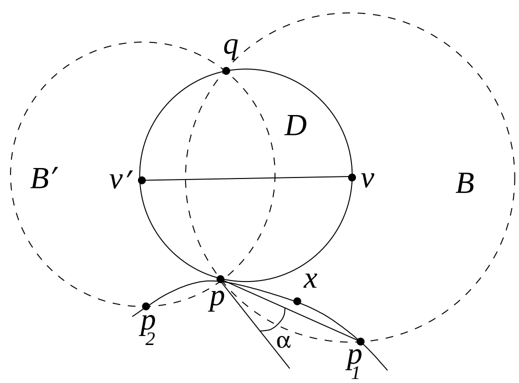

Figure 2.6: Illustration for Incorrect Edge Lemma

respectively. Therefore, the tangents make an angle of $\pi/2$. This implies that Σ cannot be tangent to both B and B' at p.

First consider the case where Σ is tangent neither to B nor to B' at p. Let p_1 and p_2 be the points of intersection of Σ with the boundaries of B and B' respectively that are consecutive to p among all such intersections. The curve segment between p and p_1 and the curve segment between p and p_2 do not have any sample point other than p. By Small Segment Lemma (2.2) both $\|p - p_1\|$ and $\|p - p_2\|$ are no more than $\frac{2\varepsilon}{1-\varepsilon} f(p)$. So, by Segment Angle Lemma (2.4) $\angle p_1 p p_2 \leq \pi - 2 \arcsin \frac{\varepsilon}{1-\varepsilon}$.

Without loss of generality, let the angle between pp_1 and the tangent to B at p be larger than the angle between pp_2 and the tangent to B' at p. Then, pp_1 makes an angle α with the tangent to B at p where

$$\begin{aligned} \alpha &\geq \frac{1}{2}\left((\pi - 2\arcsin\frac{\varepsilon}{1-\varepsilon}) - \frac{\pi}{2}\right) \\ &= \frac{\pi}{4} - \arcsin\frac{\varepsilon}{1-\varepsilon}. \end{aligned}$$

Consider the other case where Σ is tangent to one of the two balls B and B' at p. Without loss of generality, assume that it is tangent to B' at p. Then, again the lower bound on the angle α as stated above holds.

Let x be the point where the perpendicular bisector of pp_1 intersects the curve segment between p and p_1. Clearly, x is in B. Since B intersects Σ at p and q which are not consecutive sample points, it cannot contain $\gamma(p,q)$ or $\gamma'(p,q)$ inside completely. This means $B \cap \Sigma$ cannot be a 1-ball. So, by Feature Ball Lemma (1.1) B has a medial axis point and thus its radius r is at least $f(x)/2$. By simple geometry, one gets that

$$\begin{aligned} \|p - x\| &\geq \frac{1}{2}\|p - p_1\| \\ &= r \sin\alpha \\ &\geq \frac{1}{2} f(x) \sin\alpha. \end{aligned}$$

By property (iii) of Empty Segment Lemma (2.1) $\|p-x\| \leq \varepsilon f(x)$. We reach a contradiction if

$$2\varepsilon < \sin\left(\frac{\pi}{4} - \arcsin\frac{\varepsilon}{1-\varepsilon}\right).$$

For $\varepsilon < \frac{1}{5}$, this inequality is satisfied. □

Combining Correct Edge Lemma (2.6) and Incorrect Edge Lemma (2.7) we get the following theorem.

Theorem 2.1 *For $\varepsilon < \frac{1}{5}$,* CRUST *computes only all the correct edges.*