Chapter 2

Curve Reconstruction

We will describe two algorithms for curve reconstruction, CRUST and NN-CRUST in this chapter. First, we will develope some general results that will be applied to prove the correctness of the both algorithms.

A single curve in the plane is defined by a map $\xi\colon [0,1] \to \mathbb{R}^2$ where $[0,1]$ is the closed interval between 0 and 1 on the real line. The function ξ is one-to-one everywhere except at the endpoints where $\xi(0) = \xi(1)$. The curve is *smooth* if ξ has a continuous non-zero first derivative in the interior of $[0,1]$ and the right derivative at 0 is same as the left derivative at 1 both being non-zero. When we refer to a curve Σ in the plane, we actually mean the image of one or more such maps. By definition Σ does not self-intersect though can have multiple components each of which is a closed curve, i.e., without any end point.

For a finite sample to be an ε-sample for some $\varepsilon > 0$, it is essential that the local feature size f is positive everywhere. While this is true for most of the smooth curves, the definition of smoothness alone cannot prohibit f from approaching zero. Indeed there are pathological examples where a smooth curve has a zero local feature size. So, we explicitly assume that Σ has strictly positive local feature size everywhere. Semi-analytic curves satisfy this property.

For any two points x, y in Σ define two curve segments, $\gamma(x,y)$ and $\gamma'(x,y)$ between x and y, i.e., $\Sigma = \gamma(x,y) \cup \gamma'(x,y)$ and $\gamma(x,y) \cap \gamma'(x,y) = \{x,y\}$. Let P be a set of sample points from Σ. We say a curve segment is *empty* if its interior does not contain any point from P. An edge connecting two sample points, say p and q, is called *correct* if any of $\gamma(p,q)$ and $\gamma'(p,q)$ is empty. In other words, p and q are two consecutive sample points on Σ. Any edge that is not correct is called *incorrect*. The goal of *curve reconstruction* is to compute a piecewise linear curve consisting of only all correct edges. In Figure 2.1(b) all solid edges are correct and dotted edges are incorrect.

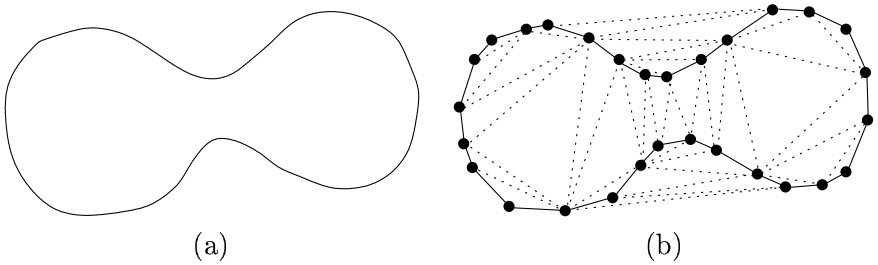

Figure 2.1: (a) A smooth curve, (b) its reconstruction from a sample shown with solid edges.

We will describe CRUST in subsection 2.2 and NN-CRUST in subsection 2.3. Some general results are presented in subsection 2.1 which are used later to claim the correctness of the algorithms.

2.1 Consequences of ε-sampling

When P is an ε-sample of Σ for sufficiently small ε, several properties can be proved.

Lemma 2.1 (Empty Segment.) *Let $p \in P$ and $x \in \Sigma$ so that $\gamma(p,x)$ is empty. Let the perpendicular bisector of px intersect the empty segment $\gamma(p,x)$ at z. For $\varepsilon < 1$, the ball $B_{z,\|p-z\|}$ has the following properties:*

(i) it intersects Σ only in $\gamma(p,x)$,

(ii) it is empty,

(iii) $\|p-z\| \leq \varepsilon f(z)$.

PROOF. Suppose $B = B_{z,\|p-z\|}$ does not intersect Σ in $\gamma = \gamma(p,x)$, see Figure 2.2. Shrink B continuously centering z till $B \cap \gamma$ becomes a 1-ball and it is tangent to some other point of Σ. Call the deformed ball B as B'. The ball B' exists as B would eventually intersect Σ in an arbitrarily small neighborhood of z which is a 1-ball and $B \cap \Sigma$ is not the 1-ball γ to begin with. The ball B' is empty of any sample point as Int B' intersects Σ only in a subset of γ which is empty. But, since $B' \cap \Sigma$ is not a 1-ball, it contains a medial axis point by Feature Ball Lemma (1.1). Then, its radius is at least $f(z)$. The point z does not have any sample point within $f(z)$ distance as B' is empty. This contradicts the ε-sampling condition for $\varepsilon < 1$. Therefore, B intersects Σ only in $\gamma(p, x)$ completing the proof of (i).

The property (ii) follows immediately as $\gamma(p, x)$ is empty and B intersects Σ only in $\gamma(p, x)$. By ε-sampling, the nearest sample point p to z is within $\varepsilon f(z)$ distance establishing (iii). □

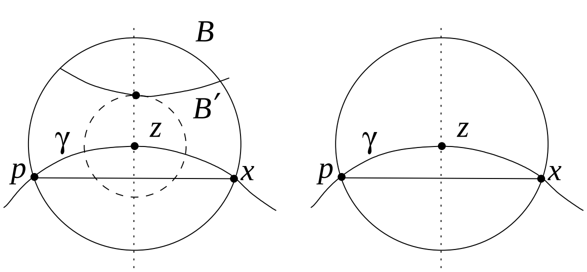

Figure 2.2: Illustration for Empty Segment Lemma. The picture on the left is impossible while the one on the right is correct.

The Empty Segment Lemma (2.1) implies that points in an empty segment are close by and any correct edge is Delaunay when ε is small.

Lemma 2.2 (Small Segment.) *Let x, y be any two points so that $\gamma(x, y)$ is empty. Then $\|x - y\| \leq \frac{2\varepsilon}{1-\varepsilon} f(x)$ for $\varepsilon < 1$.*

PROOF. Since $\gamma(x, y)$ is empty, it is a subset of an empty segment $\gamma(p, q)$ for two sample points p and q. Let z be the point where the perpendicular bisector of pq meet $\gamma(p, q)$. Consider the ball $B = B_{z,\|p-z\|}$. Since $\gamma(p, q)$ is empty, the ball B has the properties stated in the Empty Segment Lemma (2.1). Since B contains $\gamma(p, q)$, both x and y are in B. Therefore, $\|z - x\| \leq \varepsilon f(z)$ by the ε-sampling condition. By Feature Translation Lemma (1.3) $f(z) \leq \frac{f(x)}{1-\varepsilon}$. We have

$$\begin{aligned} \|x - y\| &\leq 2\|p - z\| \\ &\leq \frac{2\varepsilon}{1-\varepsilon} f(x). \end{aligned}$$

□

Lemma 2.3 (Small Edge.) *Let pq be a correct edge. Then, for $\varepsilon < 1$,*

(i) $\|p-q\| \le \frac{2\varepsilon}{1-\varepsilon} f(p)$ *for* $\varepsilon < 1$,

(ii) pq *is Delaunay.*

PROOF. Any correct edge pq has the property that either $\gamma(q,p)$ or $\gamma(p,q)$ is empty. Therefore, (i) is immediate from Small Segment Lemma (2.2). It follows from property (ii) of Empty Segment Lemma (2.1) that there exists an empty ball circumscribing the correct edge pq proving (ii). □

If three points x, y and z on Σ are sufficiently close, the segments xy and yz make small angle with the tangent at y. This implies that the angle $\angle xyz$ is close to π. As a corollary two adjacent correct edges make an angle close to π.

Lemma 2.4 (Segment Angle.) *Let x, y and z be three points on Σ with $\|x-y\|$ and $\|y-z\|$ being no more than $\frac{2\varepsilon}{1-\varepsilon} f(y)$ for $\varepsilon < 0.5$. Let α be the angle between the tangent to Σ at y and yz. Then,*

(i) $\alpha \le \arcsin \frac{\varepsilon}{1-\varepsilon}$,

(ii) $\angle xyz \ge \pi - 2\arcsin \frac{\varepsilon}{1-\varepsilon}$.

PROOF. Consider the two medial balls sandwiching Σ at y as in Figure 2.3. Let α be the angle between the tangent at y and the segment yz. Since z lies outside the medial balls, the length of the segment yz' is no more than that of yz where z' is the point of intersection of yz and a medial ball as shown.

In that case,

$$\begin{aligned} \alpha &\le \arcsin\left(\left(\frac{\|y-z'\|}{2}\right) / \left(\|m-y\|\right)\right) \\ &= \arcsin\left(\left(\frac{\|y-z\|}{2}\right) / \left(\|m-y\|\right)\right). \end{aligned}$$

It is given that $\|y-z\| \le \frac{2\varepsilon}{1-\varepsilon} f(y)$ where $\varepsilon < 0.5$. Also, $\|m-y\| \ge f(y)$ since m is a medial axis point. Plugging these values we get

$$\alpha \le \arcsin \frac{\varepsilon}{1-\varepsilon} \text{ completing the proof of (i).}$$

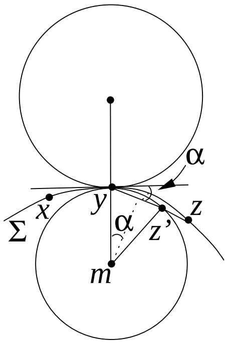

Figure 2.3: Illustration for Segment Angle Lemma.

We have

$$\begin{aligned}\angle myz &\geq \frac{\pi}{2} - \alpha \\ \angle myz &\geq \frac{\pi}{2} - \arcsin\frac{\varepsilon}{1-\varepsilon}.\end{aligned}$$

Similarly it can be shown that $\angle myx \geq \frac{\pi}{2} - \arcsin\frac{\varepsilon}{1-\varepsilon}$. The property (ii) follows immediately as $\angle xyz = \angle myz + \angle myx$. □

Since any correct edge pq has a length no more than $\frac{2\varepsilon}{1-\varepsilon}f(p)$ for $\varepsilon < 1$ (Small Edge Lemma (2.3)), we have the following result.

Lemma 2.5 (Edge Angle.) *Let pq and pr be two correct edges incident to p. We have $\angle qpr \geq \pi - 2\arcsin\frac{\varepsilon}{1-\varepsilon}$ for $\varepsilon < 1$.*