3.3 Notes and exercises

The remarkable connection between ε-samplings of a smooth surface and the Voronoi diagram of the sample points was first discovered by Amenta and Bern [AB99]. The Normal Lemma (3.2) and the Normal Variation Lemma (3.3) are two key observations made in this paper. The topological ball property that ensures the homeomorphism between the restricted Delaunay triangulation and the surface was discovered by Edelsbrunner and Shah [ES97]. Amenta and Bern showed, though not in details, that the Voronoi diagram of a sufficiently dense sample satisfies the topological ball property. The proofs presented here are adapted from Cheng, Dey, Edelsbrunner and Sullivan [CDES01].

Exercises

1. Let the restricted Voronoi cell $V_p|_\Sigma$ be adjacent to the restricted Voronoi cell $V_q|_\Sigma$ in the restricted Voronoi diagram Vor $P|_\Sigma$. Show that the distance between any two points x and y from the union of $V_p|_\Sigma$ and $V_q|_\Sigma$ is $O(\varepsilon)f(x)$ when ε is sufficiently small.

2. A version of the Edge Normal Lemma (3.4) can be derived from the Triangle Normal Lemma (3.5) albeit with a slightly worse angle bound. Derive this angle bound and carry out the proof of the topological ball property with this bound. Find out an upper bound on ε for the proof.

3. The Topological Ball Property is a sufficient condition but not necessary for the homeomorphism between a sampled surface and a restricted Delaunay triangulation of it. Establish this fact by an example.

4. Show an example where

   (i) all Voronoi edges satisfy the topological ball property, but the Voronoi cell does not,

   (ii) all Voronoi facets satisfy the topological ball property, but the Voronoi cell does not.

5. Show that for any $n > 0$, there exists a smooth surface for which a sample with n points has the Voronoi diagram where no Voronoi edge intersects the surface.

6. Let F be a Voronoi facet in the Voronoi diagram $\mathrm{Vor}\,P$ where P is an ε-sample of a smooth surface Σ. Let Σ intersect F in a single interval and the intersection points with the Voronoi edges lie within $\varepsilon f(p)$ away from p where $F \in V_p$. Show that all points of $F \cap \Sigma$ lie within $\varepsilon f(p)$ distance when ε is sufficiently small.

7. Let F and Σ be as described in exercise 6 but $F \cap \Sigma$ contains two or more topological intervals. Show that there exists a Voronoi edge $e \in F$ so that $e \cap \Sigma$ is at least $\lambda f(p)$ away from p where $\lambda > 0$ is an appropriate constant.