3.2 Topology

The sample P as a set of discrete points does not have the topology of Σ. A connection between the topology of Σ and P can be established through the restricted Voronoi and Delaunay diagrams. In particular, one can show that the underlying space of the restricted Delaunay triangulation $\mathrm{Del}\, P|_\Sigma$ is homeomorphic to Σ if the sample is sufficiently dense. Although we would not be able to compute $\mathrm{Del}\, P|_\Sigma$, the fact that it is homeomorphic to Σ will be useful in the surface reconstruction later.

3.2.1 Closed ball property

The underlying space of $\mathrm{Del}\, P|_\Sigma$ becomes homeomorphic to Σ when the Voronoi diagram $\mathrm{Vor}\, P$ intersects Σ nicely. This condition is formalized by the topological ball property which says that the restricted Voronoi cells in each dimension is a ball.

Definition 3.1 *Let F denote any Voronoi face of dimension k, $0 \leq k \leq 3$, in $\mathrm{Vor}\, P$ which intersects Σ, and $F|_\Sigma$ be the corresponding restricted Voronoi face. The pair (P, Σ) satisfies the closed ball property if $F|_\Sigma$ is a (i) $(k-1)$-ball and (ii) $\mathrm{Int}\, F \cap \Sigma = \mathrm{Int}\, F|_\Sigma$.*

The topological ball property means that Σ intersects a Voronoi cell in one disk, a Voronoi facet in one curve segment, a Voronoi edge in one point and does not intersect any Voronoi vertex. Further, there is no tangential intersection between a Voronoi face and Σ, see Figure 3.7.

The following theorem is an important result relating the topology of a surface to a point sample.

Theorem 3.1 *If the pair (P, Σ) satisfies the closed ball property, the underlying space of $\mathrm{Del}\, P|_\Sigma$ becomes homeomorphic to Σ.*

Our aim is to show that, when P is a dense sample, the topology of Σ can be captured from P. Specifically, we prove that the pair (P, Σ) satisfies the closed ball property when ε is sufficiently small. The proof frequently exploits a contradiction between the next two lemmas; the first one says that the points in a restricted Voronoi cell cannot be far apart while the second one says that any line almost normal to the surface cannot intersect it within a small distance.

Lemma 3.6 (Short Distance.) *For any two points x and y in the restricted Voronoi cell $V_p|_\Sigma$ we have*

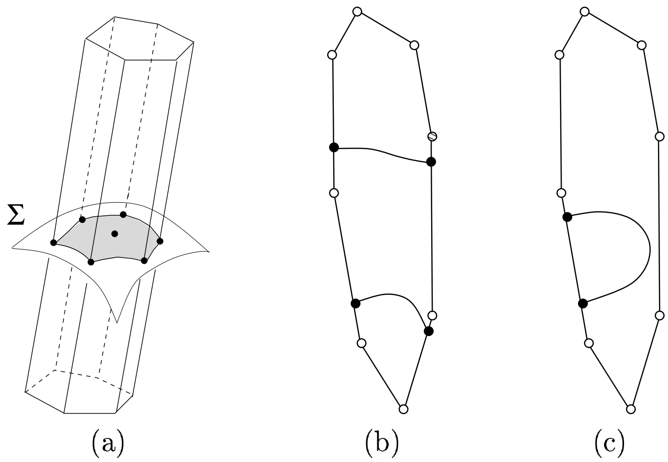

Figure 3.7: (a) A surface Σ intersects a Voronoi cell with the closed ball property, (b) intersection of Σ with a Voronoi facet is not a 1-ball, (c) intersection of Σ with a Voronoi edge is not a 0-ball.

(i) $\|x-p\| \le \frac{\varepsilon}{1-\varepsilon} f(p)$, *and*

(ii) $\|x-y\| \le \frac{2\varepsilon}{1-\varepsilon} f(x)$.

PROOF. Since x has p as the nearest sample point, $\|x-p\| \le \varepsilon f(x)$. Apply Feature Translation Lemma (1.3) to claim (i). For (ii), observe that

$$\begin{aligned} \|x-y\| &\le \|x-p\| + \|y-p\| \\ &\le \varepsilon(f(x)+f(y)) \end{aligned}$$

By Lipschitz Continuity Lemma (1.2)

$$\begin{aligned} f(y) &\le f(x) + \|x-y\| \\ &\le f(x) + \varepsilon(f(x)+f(y)), \text{ or} \\ (1-\varepsilon)f(y) &\le (1+\varepsilon)f(x). \end{aligned}$$

Therefore,

$$\|x-y\| \le \varepsilon\left(1+\frac{1+\varepsilon}{1-\varepsilon}\right) f(x) \le \frac{2\varepsilon}{1-\varepsilon} f(x).$$

□

A restricted Delaunay triangle t is dual to a Voronoi edge that intersects Σ. The intersection point, say x, belongs to a restricted Voronoi cell, say $V_p|_\Sigma$. By Short Distance Lemma (3.6), x is within $\frac{\varepsilon}{1-\varepsilon}f(p)$ distance from p. The ball $B_{x,\|x-p\|}$ circumscribes t. The diametric ball of t has a radius no more than $\|x-p\|$. Thus, the following corollary is immediate from the Short Distance Lemma (3.6).

Corollary 3.1 *The radius of the diametric ball of any restricted Delaunay triangle t is no more than $\frac{\varepsilon}{1-\varepsilon}f(p)$ where p is a vertex of t.*

Lemma 3.7 (Long Distance.) *Suppose a line intersects Σ in two points x and y and makes an angle no more than ξ with $\mathbf{n}_x$. Then $\|x-y\| \geq 2f(x)\cos\xi$.*

PROOF. Consider the two medial balls at x as in Figure 3.8. The line meets the boundaries of these two balls at x and at points that must be at least $2r\cos\xi$ distance away from x where r is the radius of the smaller of the two balls. Since $r \geq f(x)$, the result follows as y cannot lie inside the two medial balls. □

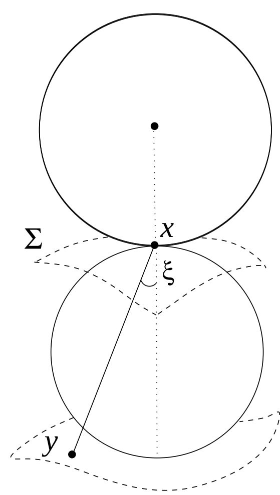

Figure 3.8: Illustration for Long Distance Lemma.