4.3 Topological guarantee

Recall that a function $h : \mathbb{X} \to \mathbb{Y}$ defines a homeomorphism between two compact Euclidean subspaces $\mathbb{X}$ and $\mathbb{Y}$ if h is continuous, one-to-one and onto. In this section, we will show a homeomorphism between Σ and any piecewise-linear 2-manifold made up of cocone triangles from T. The piecewise-linear manifold E selected by the manifold extraction step is such a space thus completing the proof of homeomorphism.

4.3.1 The map ν

We define the homeomorphism explicitly, using the function $\nu : \mathbb{R}^3 \to \Sigma$, as defined earlier. We will consider the restriction of ν to the underlying space $|E|$ of E, i.e., $\nu : |E| \to \Sigma$. Our approach will be first to show that ν is well-behaved on the sample points themselves, and then show that this behavior continues in the interior of each triangle in E.

Lemma 4.7 *For $\varepsilon \leq 0.06$, $\nu : |E| \to \Sigma$ is a continuous function.*

PROOF. By Small Triangle Lemma (4.2) every point $q \in |E|$ is within $\frac{1.15\,\varepsilon}{1-\varepsilon} f(p)$ of a triangle vertex $p \in \Sigma$ when $\varepsilon \leq 0.06$. Function ν is continuous except at the medial axis of Σ, so since $|E|$ is continuous and avoids the medial axis, ν is continuous on $|E|$. □

Let q be any point such that $\tilde{q}$ is a sample point p. By Exposed Lemma (4.6) q lies on the segment pm where m is the center of a medial ball touching Σ at p. We have the following.

Corollary 4.1 *For $\varepsilon \leq 0.06$, the function ν is one-to-one from $|E|$ to every sample p.*

In what follows, we will show that ν is indeed one-to-one on all of $|E|$. The proof proceeds in three short steps. We show that ν induces a homeomorphism on each triangle, then on each pair of adjacent triangles, and finally on $|E|$ as a whole.

Lemma 4.8 *Let U be a region contained within one triangle $t \in E$ or in adjacent triangles of E. For $\varepsilon \leq 0.06$, the function ν defines a homeomorphism between U and $\tilde{U} \subset \Sigma$.*

PROOF. We know that ν is well-defined and continuous on U, so it only remains to show that it is one-to-one. First, we prove that if U is in one triangle t, ν is one-to-one. For a point $q \in t$, the vector $\mathbf{n}_q$ from $\tilde{q}$ to q

is perpendicular to the surface at $\tilde{q}$; since Σ is smooth, the direction of $\mathbf{n}_q$ is unique and well defined. If there was some $y \in t$ with $\tilde{y} = \tilde{q}$, then q, $\tilde{q}$ and y would all be collinear and t itself would have to contain the line segment between q and y, see Figure 4.7. This implies that the normal $\mathbf{n}_q$ is parallel to the plane of t. In other words, $\mathbf{n}_q$ is orthogonal to the normal of t, contradicting the Cocone Triangle Normal Lemma (4.3) which says that the normal of t is nearly parallel to $\mathbf{n}_q$.

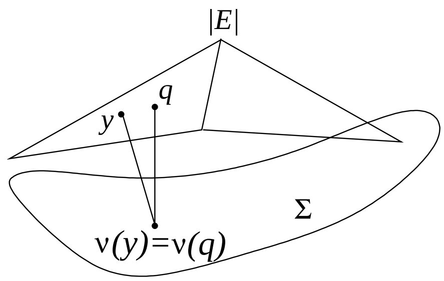

Figure 4.7: ν maps y and q to the same point which is impossible.

Now, we consider the case in which U is contained in more than one triangle. Let q and y be two points in U such that $\tilde{q} = \tilde{y} = x$, and let v be a common vertex of the triangles that contain U. Since ν is one-to-one in one triangle, q and y must lie in the two distinct triangles t_q and t_y. The line l through x with direction $\mathbf{n}_x$ pierces the patch U at least twice; if y and q are not adjacent intersections along l, redefine q so that this is true ($\tilde{q} = x$ for any intersection q of l with U). Now consider the orientation of the patch U according to the direction to the pole at v. Either l passes from inside to outside and back to inside when crossing y and q, or from outside to inside and back to outside.

The acute angles between the triangle normals of t_q, t_y and $\mathbf{n}_x$ are less than $42°$, that is, the triangles are stabbed nearly perpendicularly by $\mathbf{n}_x$. But since the orientation of U is opposite at the two intersections, the angle between the two *oriented* triangle normals is greater than zero, meaning that t_q and t_y must meet at v at an acute angle. This would contradict PROPERTY I, which is that t_q and t_y meet at v at an obtuse angle. Hence there are no two points in y, q with $\tilde{q} = \tilde{y}$.

4.3.2 Homeomorphism proof

We finish the proof for homeomorphism guarantee using a theorem from topology.

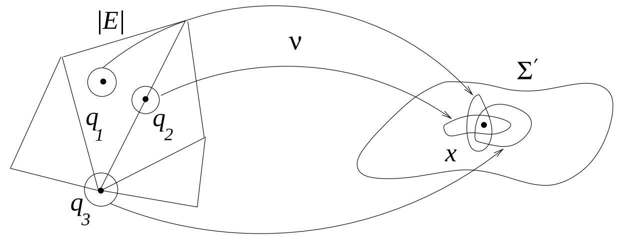

Figure 4.8: Proof of Homeomorphism Theorem.

Theorem 4.3 (Homeomorphism.) *The map ν defines a homeomorphism from the surface $|E|$ computed by* COCONE *to the surface Σ for $\varepsilon \leq 0.06$.*

PROOF. Let $\Sigma' \subset \Sigma$ be $\nu(|E|)$. We first show that $(|E|, \nu)$ is a *covering space* of Σ'. Informally, $(|E|, \nu)$ is a covering space for Σ' if function ν maps $|E|$ smoothly onto Σ', with no folds or other singularities. Showing that $(|E|, \nu)$ is a covering space is weaker than showing that ν defines a homeomorphism, since, for instance, it does not preclude several connected components of $|E|$ mapping onto the same component of Σ', or more interesting behavior, such as a torus wrapping twice around another torus to form a *double covering*.

Formally, the $(|E|, \nu)$ is a covering space of Σ' if, for every $x \in \Sigma'$, there is a path-connected *elementary neighborhood* V_x around x such that each path-connected component of $\nu^{-1}(V_x)$ is mapped homeomorphically onto V_x by ν.

To construct such an elementary neighborhood, note that the set of points $|\nu^{-1}(x)|$ corresponding to a point $x \in \Sigma'$ is non-zero and finite, since ν is one-to-one on each triangle of E and there are only a finite number of triangles. For each point $q \in \nu^{-1}(x)$, we choose an open neighborhood U_q of q, homeomorphic to a disk and small enough so that U_q is contained only in triangles that contain q.

We claim that ν maps each U_q homeomorphically onto $\tilde{U_q}$. This is because it is continuous, it is onto $\tilde{U_q}$ by definition, and, since any two points x and y in U_q are in adjacent triangles, it is one-to-one by Lemma 4.8.

Let $U'(x) = \cap_{q \in \nu^{-1}(x)} \nu(U_q)$, the intersection of the maps of each of the U_q. $U'(x)$ is the intersection of a finite number of open neighborhoods, each containing x, so we can find an open disk V_x around x. V_x is path connected, and each component of $\nu^{-1}(V_x)$ is a subset of some U_q and hence is mapped homeomorphically onto V_x by ν. Thus $(|E|, \nu)$ is a covering space for Σ'.

We now show that ν defines a homeomorphism between $|E|$ and Σ'. Since $\nu\colon |E| \to \Sigma'$ is onto by definition, we need only that ν is one-to-one. Consider one connected component G of Σ'. A theorem of algebraic topology says that when $(|E|, \nu)$ is a covering space of Σ', the sets $\nu^{-1}(x)$ for all $x \in G$ have the same cardinality. We now use Corollary 4.1, that ν is one-to-one at every sample. Since each connected component of Σ contains some samples, it must be the case that ν is everywhere one-to-one, and $|E|$ and Σ' are homeomorphic.

Finally, we show that $\Sigma' = \Sigma$. Since $|E|$ is closed and compact, Σ' must be as well. So Σ' cannot include part of a connected component of Σ, and hence Σ' must consist of a subset of the connected components of Σ. Since every connected component of Σ contains a sample p (actually many samples), and $\nu(p) = p$, all components of Σ belong to Σ', $\Sigma' = \Sigma$, and $|E|$ and Σ are homeomorphic. □

It can also be shown that $|E|$ and Σ are isotopic. We will show a technique to prove isotopy in section 6.1.3.