4.2 Geometric guarantees

In this section we establish more properties of the cocone triangles which are eventually used to prove the geometric and topological guarantees of the output of COCONE. We introduce a map $\nu : \mathbb{R}^3 \to \Sigma$ that takes each point $x \in \mathbb{R}^3$ to its closest point in Σ. This map will be used at many places in this chapter and the chapters to follow. Let

$$\begin{aligned} \tilde{x} &= \nu(x) \text{ for any point } x \in \mathbb{R}^3 \text{ and} \\ \tilde{U} &= \{\tilde{x} : x \in U\} \text{ for any set } U \subset \mathbb{R}^3. \end{aligned}$$

First, we show that all points of the cocone triangles lie close to the surface. This, in turn, allows us to extend the Cocone Triangle Normal Lemma (4.3) to the interior points of the cocone triangles. The restriction of ν to the underlying space $|T|$ of the set of cocone triangles T is a well-defined function $\nu : |T| \to \Sigma$. For if some point x had more than one closest point on the surface when $\varepsilon \le 0.06$, x would be a point of the medial axis giving $\|p - x\| \ge f(p)$ for any vertex p of a triangle in T; but by Small Triangle Lemma (4.2) every point $x \in T$ is within $\frac{1.15\,\varepsilon}{1-\varepsilon} f(p)$ of a triangle vertex $p \in \Sigma$ for $\varepsilon \le 0.06$.

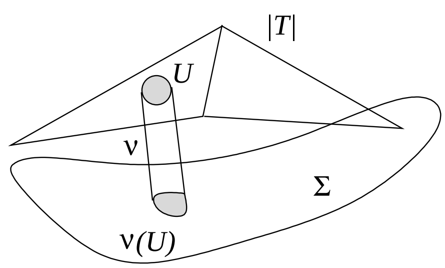

Figure 4.5: The map ν between $|T|$ and Σ.

$\tilde{O}(\varepsilon)$ **notation**: In the next two lemmas and also later we use the notation $\tilde{O}(\varepsilon)$. A value is $\tilde{O}(\varepsilon)$ if there exist two constants $\varepsilon_0 > 0$ and $c > 0$ so that the value is less than $c\varepsilon$ for any positive ε less than or equal to ε_0.

Lemma 4.4 *Let q be any point in a triangle $t \in T$. The distance between q and the point $\tilde{q}$ is $\tilde{O}(\varepsilon) f(\tilde{q})$ and is at most* $0.088\, f(\tilde{q})$ *for* $\varepsilon \le 0.06$.

PROOF. By Small Triangle Lemma (4.2) the circumradius of t is at most $\mu f(p)$ where $\mu = \frac{1.15\,\varepsilon}{1-\varepsilon} \le .074$ and p is any of its vertices. Let p be a vertex of t subtending a maximal angle of t. Since there is a sample point, namely a vertex of t, within $\mu f(p)$ distance from q, we have $\|q - \tilde{q}\| \leq \mu f(p)$. We are interested in expressing this bound in terms of $f(\tilde{q})$, so we need an upper bound on $\|p - \tilde{q}\|$.

The triangle vertex p has to lie outside the medial ball at $\tilde{q}$, while, since $\tilde{q}$ is the nearest surface point to q, q must lie on the segment between $\tilde{q}$ and the center of this medial ball. For any fixed $\|p - q\|$, these facts imply that $\|p - \tilde{q}\|$ is maximized when the angle $pq\tilde{q}$ is a right angle. Thus, $\|p - \tilde{q}\| \leq \sqrt{5}\mu f(p) \leq 0.17\ f(p)$ for $\varepsilon \leq 0.06$. This implies that $f(p) = \tilde{O}(\varepsilon) f(\tilde{q})$ and in particular $f(p) \leq 1.20\ f(\tilde{q})$ by Lipschitz property of f, giving $\|p - \tilde{q}\| = \tilde{O}(\varepsilon) f(\tilde{q})$ and $\|p - q\| \leq 0.20\ f(\tilde{q})$. We have $\|q - \tilde{q}\| \leq \mu f(p) = \tilde{O}(\varepsilon) f(\tilde{q})$ and $\|q - \tilde{q}\| \leq 0.088\ f(\tilde{q})$ in particular. □

With a little more work, we can also show that the triangle normal agrees with the surface normal at $\tilde{q}$.

Lemma 4.5 *Let q be a point on triangle $t \in T$. The angle $\angle(\mathbf{n}_{\tilde{q}}, \mathbf{n}_p)$ is at most $28°$ where p is a vertex of t with a maximal angle. Also, the angle $\angle_a(\mathbf{n}_{\tilde{q}}, \mathbf{n}_t)$ is $\tilde{O}(\varepsilon)$ and is at most $42°$ for $\varepsilon \leq 0.06$.*

PROOF. We have already seen in the proof of Lemma 4.4 that $\|p - \tilde{q}\| = \tilde{O}(\varepsilon) f(p)$. In particular, $\|p - \tilde{q}\| \leq 0.17 f(p)$ when $\varepsilon \leq 0.06$. Applying Normal Variation Lemma (3.3), and taking $\rho = \tilde{O}(\varepsilon)$ and $\rho = 0.17$, shows that the angle between $\mathbf{n}_{\tilde{q}}$ and $\mathbf{n}_p$ is $\tilde{O}(\varepsilon)$ and is less than $28°$. The angle between $\mathbf{n}_t$ and $\mathbf{n}_p$ is $\tilde{O}(\varepsilon)$ and is less than $14°$ for $\varepsilon \leq 0.06$ by Cocone Triangle Normal Lemma (4.3). Thus, the triangle normal and $\mathbf{n}_{\tilde{q}}$ make $\tilde{O}(\varepsilon)$ angle which is at most $42°$ for $\varepsilon \leq 0.06$.

Lemma 4.2, Lemma 4.4 and Lemma 4.5 imply that the output surface $|E|$ of COCONE is close to Σ both point-wise and normal-wise. The following theorem states this precisely.

Theorem 4.2 *The surface $|E|$ output by* COCONE *satisfies the following geometric properties for $\varepsilon \leq 0.06$:*

(i) Each point $p \in |E|$ is within $\tilde{O}(\varepsilon) f(x)$ distance of a point $x \in \Sigma$. Conversely, each point $x \in \Sigma$ is within $\tilde{O}(\varepsilon) f(x)$ distance of a point in Σ.

(ii) Each point p in a triangle $t \in E$ satisfies $\angle_a(\mathbf{n}_{\tilde{p}}, \mathbf{n}_t) = \tilde{O}(\varepsilon)$.

4.2.1 Additional properties

We argued in section 4.1.4 that the underlying space of the simplicial complex output by COCONE is a 2-manifold. Let E be this simplicial complex output by COCONE. A pair of triangles $t_1, t_2 \in E$ are *adjacent* if they share at least one common vertex p. Since the normals to all triangles sharing p differ from the surface normal at p by at most $42°$ (apply Triangle Normal Lemma (3.5)), and that normal in turn differs from the pole vector at p by less than $8°$ (apply Pole Lemma (4.1)), we can orient the triangles sharing p, arbitrarily but consistently. We call the normal facing the positive pole the *inside* normal and the normal facing away from it the *outside* normal. Let θ be the angle between the two inside normals of t_1, t_2. We define the angle at which the two triangles meet at p to be $\pi - \theta$.

PROPERTY I: Every two adjacent triangles in E meet at their common vertex at an angle greater than $\pi/2$.

Requiring this property excludes manifolds which contain sharp folds and, for instance, flat tunnels. Since the cocone triangles are all nearly perpendicular to the surface normals at their vertices (Cocone Triangle Normal Lemma (4.3)), and the manifold extraction step eliminates triangles adjacent to sharp edges, E has this property.

PROPERTY II: Every point in P is a vertex of E.

Restricted Delaunay Theorem (4.1) ensures that the set T of cocone triangles contains the restricted Delaunay triangulation even after the pruning. Therefore at this point T contains a triangle adjacent to every sample point in P. Lemma 4.6 below says that each sample point is exposed to the outside for some component of T to which it belongs. This ensures that at least one triangle is selected for each sample point by the manifold extraction step. This implies that E has the second property as well.

Lemma 4.6 (Exposed.) *Let p be a sample point and let m be the center of a medial ball B tangent to Σ at p. No cocone triangle intersects the interior of the segment pm for $\varepsilon \leq 0.06$.*

PROOF. In order to intersect the segment pm, a candidate triangle t would have to intersect B, and so would the smallest empty ball D circumscribing t. Let H be the plane of the circle where the boundaries of B and D intersect. See Figure 4.6. We argue that H separates the interior of pm and t.

On one side of H, B is contained in D, and on the other, D is contained in B. Since the vertices of t lie on Σ and hence not in the interior of B, t has

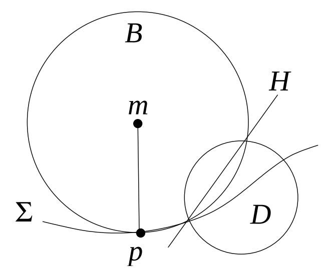

Figure 4.6: Illustration for the Exposed Lemma.

to lie in the open halfspace, call it H^+, in which D is outside B. Since D is empty, p cannot lie in the interior of D; but since p lies on the boundary of B, it therefore cannot lie in H^+. We claim that $m \notin H^+$ either.

Since $m \in B$, if it lies in H^+ then m would have to be contained in D. Since m is a point of the medial axis, this would mean that the radius of D would be at least $1/2\ f(p')$ for any vertex p' of t. For $\varepsilon \le 0.06$, this contradicts Small Triangle Lemma (4.2). Therefore p, m and hence the segment pm cannot lie in H^+, and H separates t and pm.