Lecture 9: More on curve modeling [1]

Truncating

Sometimes we need to truncate a curve at certain places for modeling purpose. This would require to compute the new control points that generate the truncated piece of the curve. We can use the matrix based subdivision to perform this truncation operation.

Truncating Bézier curves

We assume the case of a second degree Bézier curve. Recall that a second degree Bézier curve is given by three control points, $\mathbf{p}_0, \mathbf{p}_1$ and $\mathbf{p}_2$ and the relevant matrix M is:

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Suppose we want to truncate the curve between $u = u_1$ and $u = u_2$. Let $\Delta u = u_2 - u_1$. We want the curve $\mathbf{p}(v)$ with $v = u_1 + u.\Delta u$ where $u \in [0, 1]$. So, we write

$$\begin{aligned}
\mathbf{p}(v) &= \mathbf{VMP} \\
&= \left[v^2 \ v \ 1\right] \mathbf{MP} \\
&= \left[(u_1 + u.\Delta u)^2 \ (u_1 + u.\Delta u) \ 1\right] \mathbf{MP} \\
&= \left[u^2 \ u \ 1\right] \mathbf{XMP} \\
&= \mathbf{UMM}^{-1}\mathbf{XMP} \\
&= \mathbf{UMSP} \\
&= \mathbf{UMQ}
\end{aligned}$$

where $\mathbf{S} = \mathbf{M}^{-1}\mathbf{XM}$ and $\mathbf{X}$ is the matrix such that $UX = V$.

One can derive the matrix $\mathbf{S}$ in terms of u_1 and u_2. Then we can get the new control point vector $\mathbf{Q} = [\mathbf{q}_0 \ \mathbf{q}_1 \ \mathbf{q}_2]$. It can be found out that

$$\begin{aligned}
\mathbf{q}_0 &= (1 - u_1)^2\mathbf{p}_0 + 2u_1(1 - u_1)\mathbf{p}_1 + u_1^2\mathbf{p}_2 \\
\mathbf{q}_1 &= (1 - u_1)(1 - u_2)\mathbf{p}_0 + (-2u_1u_2 + u_2 + u_1)\mathbf{p}_1 + u_1u_2\mathbf{p}_2 \\
\mathbf{q}_2 &= (1 - u_2)^2\mathbf{p}_0 + 2u_2(1 - u_2)\mathbf{p}_1 + u_2^2\mathbf{p}_2
\end{aligned}$$

.

One can apply this technique to a Bézier curev of any degree.

[1] Note by Tamal K. Dey

Closed curves

We consider the case with B-splines. Uniform B-splines are suitable for generating the closed curves. The control polygon in this case is a closed polygon. For a second degree B-spline closed curve, the curve segments are written as:

$$\mathbf{p}_i(u) = \mathbf{UM}\begin{bmatrix} \mathbf{p}_{(i-1)mod(n+1)} \\ \mathbf{p}_{imod(n+1)} \\ \mathbf{p}_{(i+1)mod(n+1)} \end{bmatrix}$$

See the pictures in page 132 for closed cubic B-spline curves. The control polygon can self-intersect. In that case the curve will also self-intersect. If one pulls a control point, the curve is pulled locally twowards the control point. Again, see the picture in page 133. If one uses multiple control points, the curve becomes sharper and sharper near that control point.

Continuity

A second degree B-spline closed curve is C^1 continuous. Similarly, a cubic B-spline is C^2 continuous. We already know that the curve segments individually follow this property. But, we need to be sure that they join with the proper continuity. Lets take the case of a cubic B-spline. We know

$$\mathbf{p}_i(u) = N_{1,4}(u)\mathbf{p}_{i-1} + N_{2,4}(u)\mathbf{p}_i + N_{3,4}(u)\mathbf{p}_{i+1} + N_{4,4}(u)\mathbf{p}_{i+2}$$

where

$$\begin{aligned} N_{1,4} &= 1/6(-u^3 + 3u^2 - 3u + 1) \\ N_{2,4}(u) &= 1/6(3u^3 - 6u^2 + 4) \\ N_{3,4}(u) &= 1/6(-3u^3 + 3u^2 + 3u + 1) \\ N_{4,4} &= 1/6u^3 \end{aligned}$$

One can observe that

$$\begin{aligned} \mathbf{p}_i(1) &= 1/6(\mathbf{p}_i + 4\mathbf{p}_{i+1} + \mathbf{p}_{i+2}) = \mathbf{p}_{i+1}(0) \\ \mathbf{p}_i^u(1) &= 1/2(-\mathbf{p}_i + \mathbf{p}_{i+2}) = \mathbf{p}_{i+1}^u(0) \\ \mathbf{p}_i^{uu}(1) &= \mathbf{p}_i - 2\mathbf{p}_{i+1} + \mathbf{p}_{i+2} = \mathbf{p}_{i+1}^u u(0) \end{aligned}$$

This shows the C^2 continuity of the cubic B-splines.