Lecture 8: Matrix based Subdivisions [1]

Matrix form for B-splines

We saw that a second degree uniform B-spline curve is given by $\mathbf{p}_i(u) = \mathbf{UMP}$ for $i \in [1, n-1]$ and $0 \le u < 1$ where

$$\mathbf{M} = \frac{1}{2}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

and $\mathbf{P} = [\mathbf{p}_{i-1}\ \mathbf{p}_i\ \mathbf{p}_{i+1}]^T$.

To produce a binary subdivision we will produce two curves $\mathbf{p}_{i[0,1/2]}(u)$ and $\mathbf{p}_{i[1/2,1]}(u)$, which split the curve $\mathbf{p}_i(u)$ at $u = 1/2$. By reparameterization we can write $\mathbf{p}_{i[0,1/2]}(u)$ as $\mathbf{p}_i(u/2)$. So,

$$\begin{aligned}
\mathbf{p}_i(u/2) &= [\frac{u^2}{4}\ \frac{u}{2}\ 1]\mathbf{MP} \\
&= [u^2\ u\ 1]\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{MP} \\
&= \mathbf{UMM^{-1}XMP} \\
&= \mathbf{UMS}_{[0,1/2]}\mathbf{P} \\
&= \mathbf{UMQ}
\end{aligned}$$

where

$$\mathbf{X} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\mathbf{Q} = \begin{bmatrix} \mathbf{q}_{i-1} \\ \mathbf{q}_i \\ \mathbf{q}_{i+1} \end{bmatrix} = \mathbf{S}_{[0,1/2]}\begin{bmatrix} \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \end{bmatrix}$$

with

$$\mathbf{S}_{[0,1/2]} = \mathbf{M}^{-1}\mathbf{XM}.$$

The matrix $\mathbf{S}_{[0,1/2]}$ is called a "splitting matrix". One can calculate that

$$\mathbf{S}_{[0,1/2]} = \frac{1}{4}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

[1] Note by Tamal K. Dey

Carrying out the calculations, we obtain that

$$\begin{bmatrix} \mathbf{q}_{i-1} \\ \mathbf{q}_i \\ \mathbf{q}_{i+1} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3\mathbf{p}_{i-1} + \mathbf{p}_i \\ \mathbf{p}_{i-1} + 3\mathbf{p}_i \\ 3\mathbf{p}_i + \mathbf{p}_{i+1} \end{bmatrix}$$

Observe that $\mathbf{p}_{i[0,1/2]}(u)$ is a uniform quadratic B-spline curve defined by $\mathbf{Q}$.

Second half

To produce the second half that is $\mathbf{p}_{i[1/2,1]}(u)$ we look at $\mathbf{p}_i(u/2 + 1/2)$.

$$\begin{aligned} \mathbf{p}_i(u/2 + 1/2) &= [\frac{(u+1)^2}{4} \ \frac{(u+1)}{2} \ 1]\mathbf{MP} \\ &= [u^2 \ u \ 1] \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix} \mathbf{MP} \\ &= \mathbf{UMM^{-1}YMP} \\ &= \mathbf{UMS}_{[1/2,1]}\mathbf{P} \\ &= \mathbf{UMR} \end{aligned}$$

where

$$\mathbf{Y} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{2} & 1 \end{bmatrix}$$

and

$$\mathbf{R} = \begin{bmatrix} \mathbf{r}_{i-1} \\ \mathbf{r}_i \\ \mathbf{r}_{i+1} \end{bmatrix} = \mathbf{S}_{[1/2,1]} \begin{bmatrix} \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \end{bmatrix}$$

with

$$\mathbf{S}_{[1/2,1]} = \mathbf{M}^{-1}\mathbf{YM}.$$

$$\mathbf{S}_{[1/2,1]} = \frac{1}{4} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

Carrying out the calculations, we obtain that

$$\begin{bmatrix} \mathbf{r}_{i-1} \\ \mathbf{r}_i \\ \mathbf{r}_{i+1} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} \mathbf{p}_{i-1} + 3\mathbf{p}_i \\ 3\mathbf{p}_{i-1} + \mathbf{p}_i \\ \mathbf{p}_i + 3\mathbf{p}_{i+1} \end{bmatrix}$$

Again, $\mathbf{p}_{[1/2,1]}(u)$ is a uniform B-spline curve with $\mathbf{R}$ as control point vector.

Refinement procedure

Out of the six new control points, four are unique and two are shared, $\mathbf{q}_i = \mathbf{r}_{i-1}$ and $\mathbf{q}_{i+1} = \mathbf{r}_i$. The four conrol point $\mathbf{q}_{i-1}, \mathbf{q}_i, \mathbf{r}_i, \mathbf{r}_{i+1}$ form the new control polygon. One can apply the procedure again and again to get better approximations. Since the two splitting matrices share two rows we can combine them into one to obtain the four new control points as follows:

$$\begin{bmatrix} \mathbf{q}_{i-1} \\ \mathbf{q}_i \\ \mathbf{r}_i \\ \mathbf{r}_{i+1} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{p}_{i-1} \\ \mathbf{p}_i \\ \mathbf{p}_{i+1} \end{bmatrix}$$