Lecture 19: Bicubic B-spline subdivision [1]

Matrix subdivision

We will use the matrix subdivision technique to subdivide a bicubic B-spline patch. Such a patch is defined by an array of 4×4 array of control points. We have:

$$\mathbf{p}(u,w) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \mathbf{MPM}^T \begin{bmatrix} w^3 \\ w^2 \\ w \\ 1 \end{bmatrix}$$

where $\mathbf{M}$ is a 4×4 matrix given by:

$$\mathbf{M} = \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

We use reparameterization with $u' = \frac{u}{2}$ and $w' = \frac{w}{2}$ to obtain

$$\begin{aligned}
\mathbf{p}(u',w') &= \mathbf{p}(\frac{u}{2},\frac{w}{2}) \\
&= \begin{bmatrix} \frac{u^3}{8} & \frac{u^2}{4} & \frac{u}{2} & 1 \end{bmatrix} \mathbf{MPM}^T \begin{bmatrix} \frac{w^3}{8} \\ \frac{w^2}{4} \\ \frac{w}{2} \\ 1 \end{bmatrix} \\
&= \mathbf{U} \begin{bmatrix} \frac{1}{8} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{MPM}^T \begin{bmatrix} \frac{1}{8} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \mathbf{W}^T \\
&= \mathbf{UMM}^{-1}\mathbf{XMPM}^T\mathbf{X}^T(\mathbf{M}^{-1})^T\mathbf{M}^T\mathbf{W}^T \\
&= \mathbf{UMSPS}^T\mathbf{M}^T\mathbf{W}^T \\
&= \mathbf{UMP'M}^T\mathbf{W}^T
\end{aligned}$$

where $\mathbf{P'} = \mathbf{SPS}^T$ and $\mathbf{S} = \mathbf{M}^{-1}\mathbf{XM}$. The splitting $\mathbf{S}$ is given by

$$\mathbf{S} = \frac{1}{8} \begin{bmatrix} 4 & 4 & 0 & 0 \\ 1 & 6 & 1 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 1 & 6 & 1 \end{bmatrix}$$

If we carry out the calculation for $\mathbf{P'}$, the new control point array is generated from the old ones with certain combinations. We can categorize the new points into three categories, *face points*,

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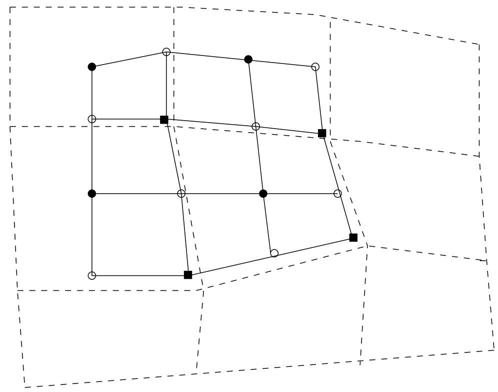

edge points, and *vertex points*. The points $\mathbf{p}'_{00}, \mathbf{p}'_{02}, \mathbf{p}'_{20}$ and $\mathbf{p}'_{22}$ are called the face points and are calculated as the average of the four points of the respective face. They are shown in the figure with solid circles. If we define F_{ij} as the average of the points $\mathbf{p}_{ij}, \mathbf{p}_{i+1,j}, \mathbf{p}_{i,j+1}$ and $\mathbf{p}_{i+1,j+1}$, then we have:

$$
\begin{aligned}
\mathbf{p}'_{00} &= F_{00} \\
\mathbf{p}'_{01} &= \frac{4F_{00} + 4F_{01} + 4\mathbf{p}_{01} + 4\mathbf{p}_{11}}{16} \\
\mathbf{p}'_{02} &= F_{01} \\
\mathbf{p}'_{03} &= \frac{4F_{01} + 4F_{02} + 4\mathbf{p}_{02} + 4\mathbf{p}_{12}}{16} \\
\mathbf{p}'_{10} &= \frac{4F_{00} + 4F_{10} + 4\mathbf{p}_{10} + 4\mathbf{p}_{11}}{16} \\
\mathbf{p}'_{11} &= \frac{4F_{00} + 4F_{01} + 4F_{10} + 4F_{11} + 4\mathbf{p}_{10} + 4\mathbf{p}_{01} + 32\mathbf{p}_{11} + 4\mathbf{p}_{21} + 4\mathbf{p}_{12}}{64} \\
\mathbf{p}'_{12} &= \frac{4F_{01} + 4F_{11} + 4\mathbf{p}_{11} + 4\mathbf{p}_{12}}{16} \\
\mathbf{p}'_{13} &= \frac{4F_{01} + 4F_{02} + 4F_{11} + 4F_{12} + 4\mathbf{p}_{11} + 4\mathbf{p}_{02} + 32\mathbf{p}_{12} + 4\mathbf{p}_{22} + 4\mathbf{p}_{13}}{64} \\
\mathbf{p}'_{20} &= F_{10} \\
\mathbf{p}'_{21} &= \frac{4F_{10} + 4F_{11} + 4\mathbf{p}_{11} + 4\mathbf{p}_{21}}{16} \\
\mathbf{p}'_{22} &= F_{11} \\
\mathbf{p}'_{23} &= \frac{4F_{11} + 4F_{12} + 4\mathbf{p}_{12} + 4\mathbf{p}_{22}}{16} \\
\mathbf{p}'_{30} &= \frac{4F_{10} + 4F_{20} + 4\mathbf{p}_{20} + 4\mathbf{p}_{21}}{16} \\
\mathbf{p}'_{31} &= \frac{4F_{10} + 4F_{20} + 4F_{11} + 4F_{21} + 4\mathbf{p}_{20} + 4\mathbf{p}_{11} + 32\mathbf{p}_{21} + 4\mathbf{p}_{31} + 4\mathbf{p}_{22}}{64} \\
\mathbf{p}'_{32} &= \frac{4F_{11} + 4F_{21} + 4\mathbf{p}_{21} + 4\mathbf{p}_{22}}{16} \\
\mathbf{p}'_{33} &= \frac{4F_{11} + 4F_{21} + 4F_{12} + 4F_{22} + 4\mathbf{p}_{21} + 4\mathbf{p}_{12} + 32\mathbf{p}_{22} + 4\mathbf{p}_{32} + 4\mathbf{p}_{23}}{64}
\end{aligned}
$$

If we examine these equations we see that the points $\mathbf{p}'_{01}, \mathbf{p}'_{03}, \mathbf{p}'_{10}, \mathbf{p}'_{12}, \mathbf{p}'_{21}, \mathbf{p}'_{23}, \mathbf{p}'_{30}$ and $\mathbf{p}'_{32}$ are generated as the average of four points- the two endpoints of the edge and the two new face points for the faces it is adjacent to. If we denote the edge points as E_{ij}, then we can write

$$E_{ij} = \frac{F_{i,j-1} + F_{ij} + \mathbf{p}_{ij} + \mathbf{p}_{i+1,j}}{4}$$

or

$$E_{ij} = \frac{F_{i-1,j} + F_{ij} + \mathbf{p}_{ij} + \mathbf{p}_{i,j+1}}{4}$$

depending on the configuration of the faces lying on the two sides of the edge. The edge points are marked as hollow circles in the Figure.

The remaining points $\mathbf{p}'_{11}, \mathbf{p}'_{13}, \mathbf{p}'_{31}$ and $\mathbf{p}'_{33}$ are called vertex points and they can be calculated as

$$\mathbf{p}'_{ij} = \frac{G + 2H + v}{4}$$

where G is the average of the four face points corresponding to the four faces adjacent to the vertex, H is the average of the midpoints of four edges incident to the vertex and v is the vertex itself. The vertex points are shown with solid squares in the Figure.