Lecture 17: Subdividing Biquadratic B-spline Surfaces [1]

Matrix equation

We will generalize the matrix subdivision technique for B-spline curves to surfaces. Let us consider the biquadratic B-spline surface. The $\mathbf{P}$ matrix in this case is a 3×3 array.

$$\mathbf{p}(u,w) = \begin{bmatrix} u^2 & u & 1 \end{bmatrix} \mathbf{MPM}^T\mathbf{W}^T$$

where

$$\mathbf{M} = \frac{1}{2}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Subdividing a surface patch

A surface patch is generated with a control polyhedron of four rectangles spanning the 3×3 array of control points. With a subdivision scheme of splitting at $u = 1/2, v = 1/2$ we create four patches out of a single patch that will be defined by 4×4 array of new control points. We express the new control points in terms of the old ones using the matrix subdivision technique. We consider the subpatch $\mathbf{p}(u,w)$ for $u \in [0,1/2]$ and $w \in [0,1/2]$. Define this surface patch as

$$\mathbf{p}'(u,w) = \mathbf{p}(\frac{u}{2}, \frac{w}{2})$$

We have

$$\begin{aligned}
\mathbf{p}'(u,w) &= \mathbf{p}(\frac{u}{2}, \frac{w}{2}) \\
&= \begin{bmatrix} \frac{u^2}{4} & \frac{u}{2} & 1 \end{bmatrix} \mathbf{MPM}^T \begin{bmatrix} \frac{w^2}{4} \\ \frac{w}{2} \\ 1 \end{bmatrix} \\
&= \mathbf{U}\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{MPM}^T \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \mathbf{W}^T \\
&= \mathbf{UMM}^{-1}\mathbf{XMPM}^T\mathbf{X}^T(\mathbf{M}^{-1})^T\mathbf{M}^T\mathbf{W}^T \\
&= \mathbf{UM}(\mathbf{M}^{-1}\mathbf{XM})\mathbf{P}(\mathbf{M}^T\mathbf{X}^T(\mathbf{M}^{-1})^T)\mathbf{M}^T\mathbf{W}^T \\
&= \mathbf{UMSPS}^T\mathbf{M}^T\mathbf{W}^T \\
&= \mathbf{UMP}'\mathbf{M}^T\mathbf{W}^T
\end{aligned}$$

where

$$\mathbf{P}' = \mathbf{SPS}^T$$

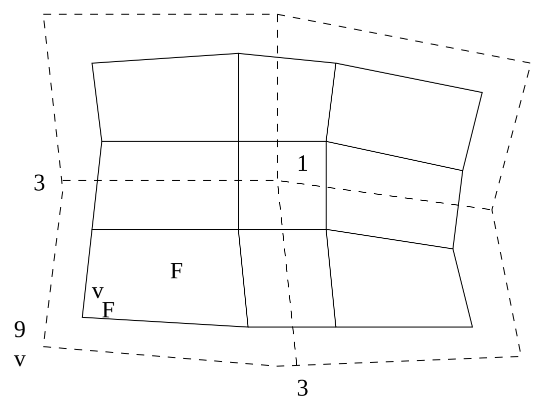
and

$$\mathbf{S} = \mathbf{M}^{-1}\mathbf{XM}$$

[1] Note by Tamal K. Dey, Ohio State U.

From the above we can conclude that $\mathbf{p}'(u,w)$ is generated with the new control points $\mathbf{P}'$. If we calculate the splitting matrix we obtain

$$\mathbf{S} = \frac{1}{4}\begin{bmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$



Using $\mathbf{P}' = \mathbf{SPS}^T$ we get the new control points as:

$$\begin{aligned}
\mathbf{p}'_{00} &= \frac{1}{16}(9\mathbf{p}_{00} + 3\mathbf{p}_{10} + 3\mathbf{p}_{01} + \mathbf{p}_{11}) \\
\mathbf{p}'_{01} &= \frac{1}{16}(3\mathbf{p}_{00} + \mathbf{p}_{10} + 9\mathbf{p}_{01} + 3\mathbf{p}_{11}) \\
\mathbf{p}'_{02} &= \frac{1}{16}(9\mathbf{p}_{01} + 3\mathbf{p}_{11} + 3\mathbf{p}_{02} + \mathbf{p}_{12}) \\
\mathbf{p}'_{10} &= \frac{1}{16}(3\mathbf{p}_{00} + 9\mathbf{p}_{10} + \mathbf{p}_{01} + 3\mathbf{p}_{11}) \\
\mathbf{p}'_{11} &= \frac{1}{16}(\mathbf{p}_{00} + 3\mathbf{p}_{10} + 3\mathbf{p}_{01} + 9\mathbf{p}_{11}) \\
\mathbf{p}'_{12} &= \frac{1}{16}(3\mathbf{p}_{01} + 9\mathbf{p}_{11} + \mathbf{p}_{02} + 3\mathbf{p}_{12}) \\
\mathbf{p}'_{20} &= \frac{1}{16}(9\mathbf{p}_{10} + 3\mathbf{p}_{20} + 3\mathbf{p}_{11} + \mathbf{p}_{21}) \\
\mathbf{p}'_{21} &= \frac{1}{16}(3\mathbf{p}_{10} + \mathbf{p}_{20} + 9\mathbf{p}_{11} + 3\mathbf{p}_{21}) \\
\mathbf{p}'_{22} &= \frac{1}{16}(9\mathbf{p}_{11} + 3\mathbf{p}_{21} + 3\mathbf{p}_{12} + \mathbf{p}_{22})
\end{aligned}$$

Observations

If we observe the above equations for new control points, we see that four points of a rectangle in the original mesh generate a new point with the weights of $9-3-3-1$. Let us call the new point v_F to denote that it is the new point corresponding to the vertex v in the rectangle F with weight 9. Then the weights of the two vertices adjacent to it will have weights 3. The opposite vertex to v in F has weight 1. Thus, each of the four vertices in F generates a new vertex v_F.

Also, one can observe that a new point can be thought of as weighing an edge in the $3-1$ ratio and then weighing these new points again with $3-1$ ratio. This is just the generalization of Chaikin's algorithm for quadratic B-spline curves.