Lecture 1: Introduction to curves

Different forms

Explicit: $y = f(x)$. Cannot represent closed or multiple valued curves.

Implicit: $f(x, y) = 0$. Represent larger class. But, difficult to compute points on the curve.

Parametric: Both explicit and implicit forms are axis dependent. It is difficult to compute points using these forms. Parametric forms are widely used to avoid these diffculties.

$x(u) = f_1(u)$ and $y(u) = f_2(u)$ for planar curves. Generally, the parametric range is normalized to $[0, 1]$ interval.

It is difficult to check membership using parametric forms.

Examples: $x = a + \ell u$, $y = b + mu$, $z = c + nu$ are parametric equations for a line in 3D. For $u = [0, 1]$, this is the line segment from $\mathbf{p}(0) = (a, b, c)$ to $\mathbf{p}(1) = ((a + \ell), (b + m), (c + n))$.

$x = u$, $y = u^2$, $z = u^3$ is a cubic curve in 3D...twisted parabola.

Curves in 3d: $F(x, y, z) = 0$ and $G(x, y, z) = 0$ represent two surfaces in 3D whose intersection is a curve. These are the implicit equations of a curve in 3D. Solving for two in terms of the other produces explicit equations.

$$y = y(x), z = z(x)$$

This can be viewed as parametric form also if we add $x = x$. The range $x = [0, 1]$ will be unacceptable. So, convert to $u = \frac{x - x_0}{x_1 - x_0}$ and $u \in [0, 1]$. Thus, a large class of implicit equations can be converted to parametric form.

Parametric space. A curve segment is $x = x(u), y = y(u), z = z(u)$ with $u \in [0, 1]$. Depending on the polynomial forms of these functions we get different types of curves, *Hermite, Bézier, and B-splines.*

A point on the curve can be thought of as a vector $\mathbf{p}(u) = (x(u), y(u), z(u))$. $\mathbf{p}^u(u) = d\mathbf{p}(u)/du = (dx(u)/du, dy(u)/du, dz(u)/du)$. These are parametric derivatives. The ordinary Cartesian derivatives are $dy/dx = (dy/du)/(dx/du)$ and similarly dy/dz and dz/dx.

The *model space* is the space where the model, i.e., the curve lives. The *parameter space* of a 3D curve are the three two dimensional spaces $(x, u), (y, u), (z, u)$.

Examples of point, straight line, plane curve and space curve.

Continuity. Geometric continuity is independent of parameterization. It is intrinsic property of the curve. Parametric continuity dependes on the parameterization. We will use G^k and C^k to denote geometric and parametric continuity respectively.

G^0 and C^0 means usual continuous curves. If the curve has continuous tangents then it is G^1 continuous. But, this does not imply equal value of the derivatives. Two curve segments $\mathbf{p}_1(u)$ and $\mathbf{p}_2(u)$ joining at a point $\mathbf{p}(u)$ is G^1 continuous if $\mathbf{p}_1^u = k_1\mathbf{p}_2^u$. But, it is C^1-continuous only if $\mathbf{p}_1^u = \mathbf{p}_2^u$ at $\mathbf{p}(u)$.

For G^2-continuity one requires *curvature continuity*, i.e., $\mathbf{p}^{uu} = k_2\mathbf{p}_2^{uu}$ at the point $\mathbf{p}(u)$. For space curves one also requires that the osculating planes coincide. Curvature of a plane curve is the inverse of the radius of the largest circle tangent at $\mathbf{p}(u)$.