Push-relabel algorithm

$O(VE^2)$

- Maintain a pre-flow.
  f satisfies skew-symmetry and capacity constraints, but not necessarily flow conservation: $f(V,u) \geq 0 \;\forall\, v \in V - \{s\}$.

- excess flow : $e(u) = f(V,u)$.
  u overflows if $e(u) > 0$.

- height $h: V \to N$, $h(s) = |V|$, $h(t) = 0$
  $h(u) \leq h(v) + 1$ for $(u,v) \in E_f$.

Push operation

- apply when $e(u) > 0$, $c_f(u,v) > 0$, and $h(u) = h(v) + 1$.

- push $d_f(u,v) = \min\{e(u), c_f(u,v)\}$ through (u,v).

- saturating push when $d_f(u,v) = c_f(u,v)$.

$\Rightarrow$ a nonsaturating push makes $e(u) = 0$.

Relabel opr.

- applies when $e(u)>0$ and $h(u) \le h(v)$ for all edges $(u,v) \in E_f$.
- Action: $h(u) = 1 + \min\{h(v) : (u,v) \in E_f\}$.

Initialization:

for $\forall v \in V$, $h(v) = 0$, $e(v) = 0$

for $\forall e \in E$, $f(u,v) = 0$, $f(v,u) = 0$

$h(s) = |V|$

for $\forall u \in Adj(s)$,

$f(s,u) = c(s,u)$, $f(u,s) = -c(s,u)$

$e(u) = c(s,u)$

$e(s) = e(s) - c(s,u)$.

$$f(u,v) = \begin{cases} c(u,v) & \text{if } u=s \\ -c(v,u) & \text{if } v=s \\ 0 & \text{otherwise} \end{cases}$$

Push-relabel (G)

1. Initialize (G, s)
2. while ∃ an applicable push or relabel
   do select an oprn and perform it.

Lemma 1 An overflowing vertex can either be pushed or relabeled.

Lemma 2 vertex heights never decreases

Lemma 3 height h maintains the property.

Lemma 4 For a preflow f, there is no path from s to t in G_f.

Theorem Push-Relabel algorithm computes max-flow if it terminates.

Proof of L1: Because of h property,

$$h(u) \le h(v)+1 \quad \forall (u,v) \in E_f.$$

If push does not apply to u,

$$h(u) < h(v)+1 \quad \forall (u,v) \in E_f.$$

Then $h(u) \le h(v) \Rightarrow$ relabling applies.

Proof of L4: By contradiction, assume

$p = \{\underset{s}{v_0}, v_1, \cdots \underset{t}{v_k}\}$ from s to t in G_f exists.

Since h is a height function,

$$h(v_i) \le h(v_{i+1})+1 \quad \text{for } i=0,1,\cdots,k-1.$$

We get $h(s) \le h(t)+k$ where $k<|V|$.

But, $h(t)=0$ and $h(s) \le k < |V|$,

contradicts $h(s)=|V|$.

Proof of Thm: Because h is maintained and L1, when the algorithm terminates, f is a flow (no excess at any vertex).

Because of L4, no path from s to t in $G_f \Rightarrow$ by max-flow-min-cut thm f is a max-flow.

Termination of Push-relable algorithm

Lemma 5: For any overflowing vertex u, there exists a simple path from u to s in G_f.

Proof. $U = \{v : \exists \text{ a simple path } u \rightsquigarrow v \text{ in } G_f\}$.

Suppose $s \notin U$ and $\bar{U} = V - U$.

- claim: $\forall v \in U,\ w \in \bar{U},\ f(w,v) \leq 0$.

  If $f(w,v) > 0 \Rightarrow f(v,w) < 0$
  $\Rightarrow c_f(v,w) > 0$
  $\Rightarrow u \rightsquigarrow v \rightarrow w$ exists in G_f
  $\Rightarrow$ contradicts $w \notin U$

- $f(\bar{U}, U) \leq 0$ (since each term nonpositive)

$$\begin{aligned} \Rightarrow e(U) &= f(V,U) \\ &= f(\bar{U},U) + f(U,U) \\ &= f(\bar{U},U) \\ &\leq 0 \end{aligned}$$

- Since $s \notin U$, $u \neq s$, and $e(u) = 0$

  $\Rightarrow$ contradicts u is overflowing.

Lemma 6: At any time of the algorithm, $h(u) \le 2|V|-1 \quad \forall u \in V$.

Proof: - Assume $u \notin \{s, t\}$ since $h(s) = |V|$, $h(t) = 0$

- Assume u is overflowing since h changes only for such vertices.

- By L5, $p = \{v_0, v_1, \cdots, v_k\}$, $v_0 = u$, $v_k = s$ is a simple path in G_f.

- Then, $h(u) = h(v_0) \le h(v_k) + k$
$\le h(s) + |V| - 1$
$= 2|V| - 1.$

Lemma 7: The # of relabling oprn is at most $2|V|-1$ per vertex, at most $(2|V|-1)(|V|) < 2|V|^2$ total

Proof: - each relable increases the height of some vertex

- height increases from 0 to at most $2|V|-1$

→ follows the claim.

⑥ tkd

Bound on saturating pushes

Lemma 8: There are at most $2|V||E|$ saturating pushes.

Proof.
- Consider saturating pushes from u to v and from v to u.
- there is an edge either uv or vu
- Two consecutive saturating pushes through (u,v) requires $h(u)$ to increase by at least 2 (why?) Same for (v,u).
- focus on $h(u)+h(v)$ for each saturating push between u and v.
- Let A be the sequence of $h(u)+h(v)$
- first push $\Rightarrow h(u)+h(v) \geq 1 \Rightarrow$ 1st # in A is at least 1.
- last push $\Rightarrow h(u)+h(v) \leq 2(2|V|-1) \leq 4|V|-2$
- Since A can have only alternate integers #saturating push between u & v $\leq 2|V|$
- Multiplying by #edges, we get at most $2|V||E|$ saturating pushes.

⑦ tkd

Bound on nonsaturating pushes

Lemma 9: There are at most $4|V|^2(|V|+|E|)$ nonsaturating pushes.

Proof: Define a potential function

$$\Phi = \sum_{v \in X} h(v) \quad \text{where } X = \text{set of overflowing vertices}$$

- Initially Φ is 0
- A relabel increases Φ by at most $2|V|$ since a vertex's height cannot be more than $2|V|$.
- A saturating push can increase Φ by at most $2|V|$ since no height changes and only v can become overflowing after the push through (u,v)
- A nonsaturating push through (u,v) ~~can~~ decreases Φ by at least 1. because u is no more overflowing and $h(v) - h(u) = -1$
- Φ increases in total by at most $2|V|(2|V|^2) + 2|V|(2|V||E|) = 4|V|^2(|V|+|E|)$
- Total decrease is at most $4|V|^2(|V|+|E|)$
- Since each nonsaturating push decreases Φ by at least 1, we have the result.

Theorem Preflow-Push algorithm terminates with $O(V^2E)$ basic opr^{n}.

Theorem Preflow-Push computes the max-flow in $O(V^2E)$ time.

Proof. Each relable can be done in $O(|V|)$ time. So total $O(|V|.(|V||E|)) = O(V^2E)$ time.

Each of $O(V^2E)$ pushes can be done in $O(1)$ time.