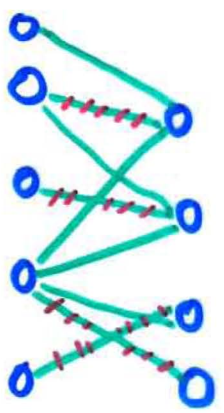
Maximum bipartite matching

Given undirected $G=(V,E)$, a matching is $M \subseteq E$ s.t. $\forall v \in V$, at most one edge in M is contains v.

Problem. Find a maximal matching in terms of #edges.

We consider maximal matching problem for bipartite graphs.



max. matching of 4

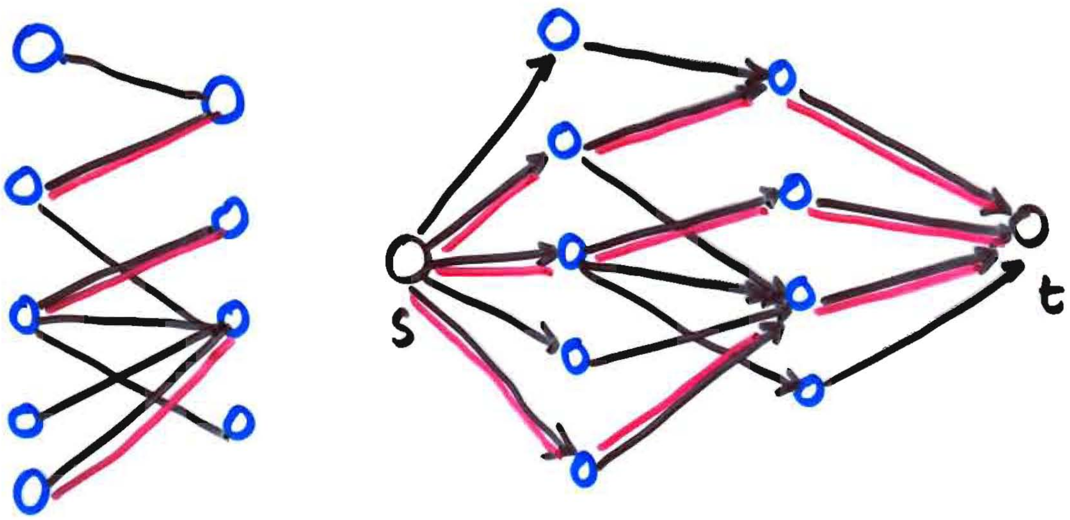
- We can use maxflow algorithm to find a max. matching in $G=(V,E)$ where $V=L\cup R$ are partitioned into two disjoint sets and all edges are between L and R.

- Define a flow network

$G' = (V', E')$; $V' = V \cup \{s, t\}$.

$E' = \{(s,u) : u \in L\}$
$\cup \{(u,v) : u \in L, v \in R, (u,v) \in E\}$
$\cup \{(v,t) : v \in R\}$.

- unit capacity to each edge.



Lemma 1 If M is a matching in G, then there is an integer-flow f in G' with $|f| = M$ and vice-versa.

Lemma 2 If capacities are integral, so is the flow (max-flow); Each $f(u,v)$ is integer.

Theorem A maximum matching M in G satisfies $|M| = |f^*|$ where f^* is the max. flow in G'.

Proof. Let f be the flow corresponding to
- M in G. Let f be not maximum.
- Then there is f' in G' s.t. $|f'| > |f|$. By Lemma 2, ~~f~~ f' is integral. Then, f' corresponds to a M' in G with

  $|M'| = |f'| > |f| = |M|$.
- Show similarly, if f is a max-flow in G', its corresponding M in G is maximum.