Duality in LP

There is a dual version of every LP. The main purpose of studying this duality is to see LP differently and prove the optimality of the Simplex algorithm

Primal:

maximize $\sum_{j=1}^{n} c_j x_j$

s.t.

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, 2, \cdots, m$$

$$x_j \geq 0, \quad j = 1, 2, \cdots, n$$

Dual:

minimize $\sum_{i=1}^{m} b_i y_i$

s.t.

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j, \quad j = 1, 2, \cdots, n$$

$$y_i \geq 0, \quad i = 1, 2, \cdots, m$$

Example.

$$\text{maximize} \quad 3x_1 + x_2 + 2x_3$$

s.t.

$$x_1 + x_2 + 3x_3 \le 30$$

$$2x_1 + 2x_2 + 5x_3 \le 24$$

$$4x_1 + x_2 + 2x_3 \le 36$$

$$x_1, x_2, x_3 \ge 0$$

$$\Downarrow$$

$$\text{minimize} \quad 30y_1 + 24y_2 + 36y_3$$

s.t.

$$y_1 + 2y_2 + 4y_3 \ge 3$$

$$y_1 + 2y_2 + y_3 \ge 1$$

$$3y_1 + 5y_2 + 2y_3 \ge 2$$

$$y_1, y_2, y_3 \ge 0$$

Lemma (Weak-duality): Let $\bar{x}$ and $\bar{y}$ be be any feasible solutions to primal and dual LP respectively. Then,

$$\sum_{j=1}^{n} c_j \bar{x}_j \le \sum_{i=1}^{m} b_i \bar{y}_i$$

Proof.

$$\sum_{j=1}^{n} c_j \bar{x}_j \le \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \bar{y}_i \right) \bar{x}_j$$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \bar{x}_j \right) \bar{y}_i$$

$$\le \sum_{i=1}^{m} b_i \bar{y}_i$$

Lemma 2 If $\sum_{j=1}^{n} c_j \bar{x}_j = \sum_{i=1}^{m} b_i \bar{y}_i$, then $\bar{x}$ and $\bar{y}$ are optimal solutions.

Proof. follows immediately from the previous Lemma.

We can read off the optimal solutions from the final slack forms. Let

$$z = v' + \sum_{j \in N} c'_j x_j$$

$$x_i = b'_i - \sum_{j \in N} a'_{ij} x_j \quad \text{for } i \in B$$

be the final slack form for primal. Then, optimal dual solution is:

$$\bar{y}_i = \begin{cases} -c'_{n+i} & \text{if } (n+i) \in N \\ 0 & \text{otherwise} \end{cases} \quad \Bigg\} \; (*)$$

In our example:

$\bar{y}_1 = 0$ (since $n+1 = 4 \in B$)

$\bar{y}_2 = -c'_5 = 1/6$

$\bar{y}_3 = -c'_6 = 2/3.$

The objective value $(30 \times 0) + (24 \times (1/6)) + (36 \times \frac{2}{3})$
$= 28$

This is equal to the optimal value that we found for the primal.

Theorem Let $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$ be a solution returned by Simplex. Let c' denote coefficients in final slack form. Let $\bar{y} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)$ be defined as in the last slide. Then $\bar{x}$ and $\bar{y}$ are optimal solutions. and

$$\sum_{j=1}^{n} c_j \bar{x}_j = \sum_{i=1}^{m} b_i \bar{y}_i. \quad \text{--- (1)}$$

Proof. We show that $\bar{x}$ and $\bar{y}$ satisfy (1).
Simplex terminates with:

$$z = v' + \sum_{j \in N} c'_j x_j$$

Since Simplex terminated,

$c_j' \leq 0$ for all $j \in N$.

Let $c_j' = 0$ for $j \in B$.

Then,

$$
\begin{aligned}
z &= v' + \sum_{j \in N} c_j' x_j \\
&= v' + \sum_{j \in N} c_j' x_j + \sum_{j \in B} c_j' x_j \\
&= v' + \sum_{j=1}^{n+m} c_j' x_j
\end{aligned}
$$

Basic sol.$^{\underline{n}}$ $\bar{x}$ with final slack form $\bar{x}_j = 0$ for all $j \in N$, and $z = v'$. Since all slack forms are equivalent:

$$
\begin{aligned}
\sum_{j=1}^{n} c_j \bar{x}_j &= v' + \sum_{j=1}^{n+m} c_j' \bar{x}_j \\
&= v' + \sum_{j \in N} c_j' \bar{x}_j + \sum_{j \in B} c_j' \bar{x}_j \\
&= v' + 0 + 0 \\
&= v'
\end{aligned}
$$

Now show $\bar{y}$ as defined is feasible, and the objective value $\sum_{i=1}^{m} b_i \bar{y}_i = \sum_{j=1}^{n} c_j \bar{x}_j$.

We have:

$$\sum_{j=1}^{n} c_j x_j = v' + \sum_{j=1}^{n+m} c_j' x_j \quad \text{(from previous deduction)}$$

$$= v' + \sum_{j=1}^{n} c_j' x_j + \sum_{j=n+1}^{n+m} c_j' x_j$$

$$= v' + \sum_{j=1}^{n} c_j' x_j + \sum_{i=1}^{m} c_{n+i}' x_{n+i}$$

$$= v' + \sum_{j=1}^{n} c_j' x_j + \sum_{i=1}^{m} (-\bar{y}_i) x_{n+i}$$

$$= v' + \sum_{j=1}^{n} c_j' x_j + \sum_{i=1}^{m} (-\bar{y}_i)\left(b_i - \sum_{j=1}^{n} a_{ij} x_j\right)$$

$$= v' + \sum_{j=1}^{n} c_j' x_j - \sum_{i=1}^{m} b_i \bar{y}_i + \sum_{j=1}^{n} \sum_{i=1}^{m} (a_{ij} \bar{y}_i) x_j$$

$$= \underbrace{\left(v' - \sum_{i=1}^{m} b_i \bar{y}_i\right)}_{\Downarrow \; 0} + \sum_{j=1}^{n} \underbrace{\left(c_j' + \sum_{i=1}^{m} a_{ij} \bar{y}_i\right)}_{\Downarrow \; c_j} x_j$$

$\Rightarrow$

$$v' - \sum_{i=1}^{m} b_i \bar{y}_i = 0$$

$$c_j' + \sum_{i=1}^{m} a_{ij} \bar{y}_i = c_j \quad \text{for } j = 1, 2, \ldots, n.$$

Therefore,

$$\sum_{i=1}^{m} b_i \bar{y}_i = v' = \sum_{j=1}^{n} c_j \bar{x}_j$$

We show $\bar{y}$ is feasible:

Since $c_j' \leq 0$ for $\forall\, j = 1, 2, \ldots, n+m$,

$$c_j = c_j' + \sum a_{ij} \bar{y}_i \qquad \text{(from previous deduction)}$$

$$\leq \sum_{i=1}^{m} a_{ij} \bar{y}_i \, . \text{ satisfying dual constraints}$$

Finally, since $c_j' \leq 0$ for $j \in NUB$, when $\bar{y}$ is set according to ~~what~~ (*), we have $\bar{y}_i \geq 0$ satisfying other constraints.

Initialization

We have to find:

1. If given LP is feasible
2. If so, an inital basic solution that is feasible.

maximize $2x_1 - x_2$

s.t.

$$2x_1 - x_2 \leq 2$$

$$x_1 - 5x_2 \leq -4$$

$$x_1, x_2 \geq 0$$

has a basic solution $x_1 = 0$ and $x_2 = 0$ which violates second constraint. So, Initialize cannot return the obvious slack form.

In order to determine feasibitity we formulate auxiliary LP.

Let L be an LP and L_{aux} be its auxiliary counterpart:

$$\begin{aligned} \text{maximize} \quad & -x_0 \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j - x_0 \le b_i \,, \quad i=1,..m \\ & x_j \ge 0 \,, \quad j=0,\dots,n. \end{aligned}$$

<u>Lemma</u> L is feasible <u>iff</u> optimal objective value of L_{aux} is 0.

<u>Proof</u>. Let $\bar{x} = (\bar{x}_1, \dots \bar{x}_n)$ be a feasible sol^{n} to L. Then $\bar{x}_0 = 0$ in combination with $\bar{x}$ is feasible for L_{aux}. This solution must be optimal for L_{aux}.

Conversely, suppose optimal objective value of L_{aux} is 0. Then, $\bar{x}_0 = 0$ and the remaining variables satisfy the constraints of L.

Initialize(A, b, c)

Let l be the index of minimum b_i

if $b_l \geq 0$ * Is the basic soln. feasible *

return $(\{1, \ldots, n\}, \{n+1, \ldots, n+m\}, A, b, c, 0)$

Form L_{aux} ; * add x_0 in constraints and object. *

Let (N, B, A, b, c, v) be the resulting slack form

$(N, B, A, b, c, v) \leftarrow \text{Pivot}(N, B, A, b, c, v, l, 0)$

requires a proof. ← * The basic solution is now feasible *

iterate while loop of Simplex until an optimal solution to L_{aux} is found;

if the basic solution sets $\bar{x}_0 = 0$,

then return the slack form with x_0 removed and the original objective function restored;

else return "infeasible".

Fundamental theorem of LP.

Any LP L, given in standard form, either

1. has an optimal solution with a final objective value
2. is infeasible, or
3. is unbounded.

If L is infeasible, Simplex returns "infeasible". If L is unbounded, Simplex returns "unbounded". Otherwise, it returns an optimal solution with an objective value.