

Linear Programming

①

A linear constraint is a linear inequality or equality.

$$a_1x_1 + a_2x_2 + \dots + a_nx_n \begin{matrix} \leq \\ = \\ \geq \end{matrix} b$$

A linear objective function is

$$c_1x_1 + c_2x_2 + \dots + c_nx_n$$

A linear program is to minimize or maximize a linear objective function under linear constraints.

$$\text{maximize } c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to

minimize

$$a_{11}x_1 + \dots + a_{1n}x_n \leq b_1$$

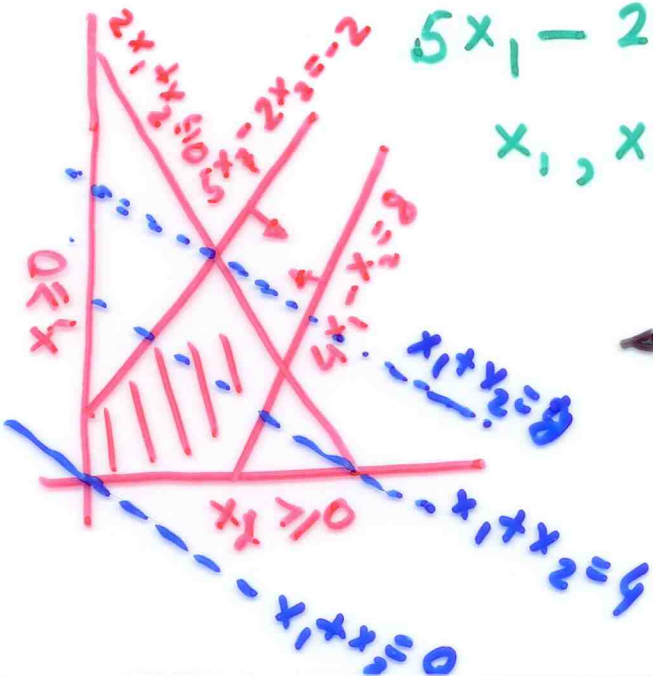
$$a_{21}x_1 + \dots + a_{2n}x_n \leq b_2$$

$$\vdots$$
$$a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m$$

- The x_i 's are called variables.
- The values of variables that satisfy all constraints constitute feasible region
- A solution to the constraints is a feasible solution
- An optimal feasible solution is the goal of linear of programming.

Ex.

Maximize $x_1 + x_2$
 S.t.
 $4x_1 - x_2 \leq 8$
 $2x_1 + x_2 \leq 10$
 $5x_1 - 2x_2 \geq -2$
 $x_1, x_2 \geq 0$



← Geometric interpretation

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- The feasible region is convex.
- The optimality happens (if at all) at a boundary points of the convex feasible region
- It is either a vertex or a line segment (in 2D), or a k -flat (in general dimension)
- Even if there are multiple optimal solutions, there is one vertex where optimality happens
- The convex region can be unbounded
 - If unbounded, the optimal solution could be infinity
 - Even if the feasible region is unbounded, the solution may not be unbounded.

Standard Form

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Given: n real numbers c_1, \dots, c_n
 m real numbers b_1, \dots, b_m
 $m \times n$ " " " $a_{ij}, \begin{matrix} i=1, \dots, m \\ j=1, \dots, n \end{matrix}$

To find: n real numbers x_1, x_2, \dots, x_n that

$$\text{maximize } \sum_{j=1}^n c_j x_j$$

s.t.

$$\sum_{j=1}^n a_{ij} x_j \leq b_i \text{ for } i=1, 2, \dots, m$$

$$x_j \geq 0 \text{ for } j=1, \dots, n.$$

The last set of constraints are called nonnegativity constraints.

Matrix form:

$$\text{maximize } c^T x$$

s.t.

$$Ax \leq b$$

$$x \geq 0$$

A : $m \times n$ matrix

b : m -dimensional vector

c : n -dimensional vector.

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Converting to standard form

1. Objective function may be minimization
2. Variables without nonnegativity constraints
3. may be equality constraints
4. greater-than-equal-to inequality constraints

We convert a LP L to a LP L' s.t.

L' is in standard form but L' is equivalent to L : each feasible solution \bar{x} to L with objective value z , there is a \bar{x}' with objective value z for L' and vice versa.

($-z$ if min-max interchanged) ←

For 1: Simply negate the objective function.
 i.e. if minimize $C^T x$ then do:
 maximize $[-C]^T x$.

For 2: if x_j appears without nonnegativity constraints, replace x_j with $x_j' - x_j''$ and add $x_j' \geq 0, x_j'' \geq 0$. Thus, replace $C_j x_j \rightarrow C_j x_j' - C_j x_j''$ and $a_{ij} x_j \rightarrow a_{ij} x_j' - a_{ij} x_j''$

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2 continued: a feasible solution \bar{x} to the new LP corresponds to a feasible solution to the original LP with

$$\bar{x}_j = \bar{x}'_j - \bar{x}''_j.$$

For 3: Convert an equality constraint as:

$$f(x) = b \Rightarrow f(x) \leq b, \text{ and } f(x) \geq b$$

Then, by (4) convert the greater-than-equal-to to less-than-equal-to.

For 4:

Convert

$$\sum_{j=1}^n a_{ij} x_j \geq b_i \Rightarrow \sum_{j=1}^n -a_{ij} x_j \leq -b_i$$

Example.

Minimize $-2x_1 + 3x_2$

S.t.

$$x_1 + x_2 = 7$$

$$x_1 - 2x_2 \leq 4$$

$$x_1 \geq 0$$

Apply 1:

Maximize $2x_1 - 3x_2$
 s.t. $x_1 + x_2 = 7$
 $x_1 - 2x_2 \leq 4$
 $x_1 \geq 0$

Apply 2:

Maximize $2x_1 - 3x_2' + 3x_2''$
 s.t. $x_1 + x_2' - x_2'' = 7$
 $x_1 - 2x_2' + 2x_2'' \leq 4$
 $x_1, x_2', x_2'' \geq 0$

Apply 3:

Maximize $2x_1 - 3x_2' + 3x_2''$
 s.t. $x_1 + x_2' - x_2'' \leq 7$
 $x_1 + x_2' - x_2'' \geq 7$
 $x_1 - 2x_2' + 2x_2'' \leq 4$
 $x_1, x_2', x_2'' \geq 0$

Apply 4:

Maximize $2x_1 - 3x_2 + 3x_3$
 s.t. $x_1 + x_2 - x_3 \leq 7$
 $-x_1 - x_2 + x_3 \leq -7$
 $x_1 - 2x_2 + 2x_3 \leq 4$
 $x_1, x_2, x_3 \geq 0$

} Standard form

Converting LP to slack form

In simplex algorithm that we will study, the constraints except the nonnegativity ones are all equalities. This is called slack form.

Let $\sum_{j=1}^n a_{ij} x_j \leq b_i$ be an inequality constraint.

Introduce a new slack variable s .

$$s = b_i - \sum_{j=1}^n a_{ij} x_j$$
$$s \geq 0$$

Instead of s , we write

$$x_{n+i} = b_i - \sum_{j=1}^n a_{ij} x_j$$
$$x_{n+i} \geq 0.$$

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Applying slack variables to the example we took, we get:

$$\text{maximize } 2x_1 - 3x_2 + 3x_3$$

s.t.

$$x_4 = 7 - x_1 - x_2 + x_3$$

$$x_5 = -7 + x_1 + x_2 - x_3$$

$$x_6 = 4 - x_1 + 2x_2 - 2x_3$$

$$x_1, x_2, \dots, x_6 \geq 0$$

Slack
form

The variables on the left side of equality constraints are called basic variables and the rest non-basic variables.

Drop "maximize", "s.t.", and write objective function with a linear equation, drop non-negative constraints (implicit)

$$z = 2x_1 - 3x_2 + 3x_3$$

$$x_4 = 7 - x_1 - x_2 + x_3$$

$$x_5 = -7 + x_1 + x_2 - x_3$$

$$x_6 = 4 - x_1 + 2x_2 - 2x_3$$

N: index set non-basic variables, $|N| = m$

B: index set of basic variables, $|B| = m$

C: objective function vector

b: constraint vector

A: constraint matrix

Thus, we can express the slack form by a tuple (N, B, A, b, c, v) : All are explained except v which represents a possible constant term in objective equation.

$$\begin{aligned}
 z &= v + \sum_{j \in N} c_j x_j \\
 x_i &= b_i - \sum_{j \in N} a_{ij} x_j \text{ for } i \in B.
 \end{aligned}
 \left. \vphantom{\begin{aligned} z \\ x_i \end{aligned}} \right\} \begin{array}{l} \text{Concise} \\ \text{Slack} \\ \text{form} \end{array}$$

Example

$$\begin{aligned}
 z &= 28 - \frac{x_3}{6} - \frac{x_5}{6} - \frac{2x_6}{3} \\
 x_1 &= 8 + \frac{x_3}{6} + \frac{x_5}{6} - \frac{x_6}{3} \\
 x_2 &= 4 - \frac{8x_3}{3} - \frac{2x_5}{3} + \frac{x_6}{3} \\
 x_4 &= 18 - \frac{x_3}{2} + \frac{x_5}{2},
 \end{aligned}$$

We have $B = \{1, 2, 4\}$, $N = \{3, 5, 6\}$

$$A = \begin{pmatrix} -1/6 & -1/6 & 1/3 \\ 8/3 & 2/3 & -1/3 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

$$b = \begin{pmatrix} 8 \\ 4 \\ 18 \end{pmatrix}, \quad c = \begin{pmatrix} -1/6 \\ -1/6 \\ -2/3 \end{pmatrix}, \quad v = 28.$$

Formulating problems with LP

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Shortest Path

Given $G = (V, E)$ with weight $w: E \rightarrow \mathbb{R}$,
a source $s \in V$ and a destination $t \in V$,
determine s.p. length $d[t] = \delta(s, t)$.

Minimize $d[t]$

s.t.

$d[v] \leq d[u] + w(u, v)$
for each $(u, v) \in E$

$d[s] = 0$

In this LP, there are $|V|$ variables $d[v]$.
There are $|E| + 1$ constraints.

? How would you formulate single-source shortest path using LP. The above was only for single-pair shortest path.

Maximum flow

$G = (V, E)$, each $u, v \in E$ has capacity $c(u, v) \geq 0$
 Source s , sink t . A flow:
 $f : V \times V \rightarrow \mathbb{R}$, satisfies three constraints.

Maximize $\sum_{v \in V} f(s, v)$

s.t.

$f(u, v) \leq c(u, v)$ for each $u, v \in V$

$f(u, v) = -f(v, u)$ for each $u, v \in V$

$\sum_{v \in V} f(u, v) = 0$ for each $u \in V - \{s, t\}$.

This LP has $|V|^2$ variables (each pair $(u, v) \in V \times V$)
 $2|V|^2 + |V| - 2$ constraints.

Can you rewrite the LP for the above
 with $O(|V| + |E|)$ constraints?

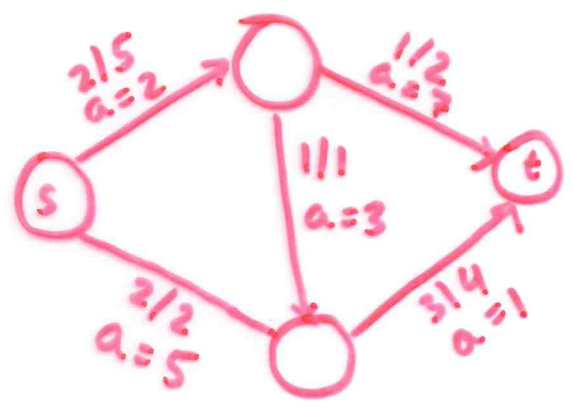
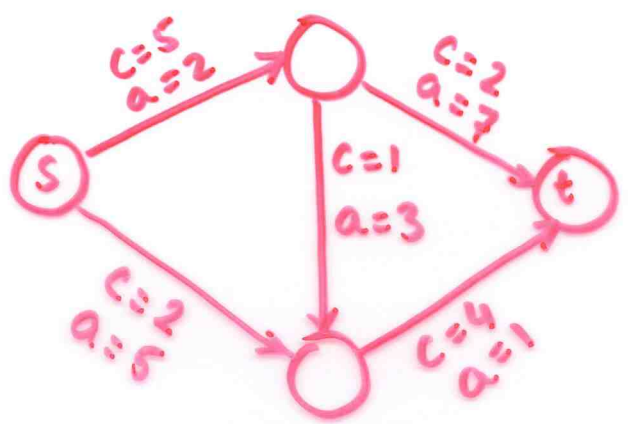
Minimum-cost flow

Each edge (u, v) has real valued cost $a(u, v)$ in addition to cost $c(u, v)$.

If $f(u, v)$ is the flow over (u, v) , we incur cost $a(u, v) f(u, v)$.

We are also given target flow d .

Goal: Send d units flow from s to t so that cost $\sum_{(u, v) \in E} a(u, v) f(u, v)$ is minimum.



Minimize $\sum_{(u, v) \in E} a(u, v) f(u, v)$

s.t.

$f(u, v) \leq c(u, v)$ for $(u, v) \in E \setminus xv$ units.

$f(u, v) = -f(v, u)$ for $(u, v) \in V \times V$

$\sum_{u \in V} f(u, v) = 0$ for $u \in V - \{s, t\}$

$\sum_{u \in V} f(s, u) = d.$

Min-cost flow with target 4

Multicommodity flow

Each edge $(u,v) \in E$ has $c(u,v) \geq 0$.

Given k commodities K_1, K_2, \dots, K_k .

$K_i = (s_i, t_i, d_i)$: specifies flow demand of d_i from source s_i to sink t_i .

- Let $f_i(u,v)$: flow over (u,v) on commodity K_i . It should satisfy the three flow constraints.

- $f(u,v) = \sum_{i=1}^k f_i(u,v) \leq c(u,v)$ (constraint)

Minimize 0

s.t. $\sum_{i=1}^k f_i(u,v) \leq c(u,v)$ for $u,v \in V \times V$
 $f_i(u,v) = -f_i(v,u)$ for $i=1, \dots, k$, and $u,v \in V \times V$

$\sum_{v \in V} f_i(u,v) = 0$ for $i=1, \dots, k$ and $u \in V - \{s_i, t_i\}$.

$\sum_{v \in V} f_i(s_i, v) = d_i$ for $i=1, \dots, k$.